DETERMINANT PART→1

Consider the two equations : $a_1x+b_1y=d_1$

$$a_2x+b_1y=d_2$$

Second order determinant : $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Row & Column Concept

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $\begin{array}{l} \rightarrow \text{Row 1} \\ \rightarrow \text{Row 2} \end{array}$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ → Column 2

↓ Column 1

Evaluation of determinant of order 2

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

Example : (a) $\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = 2\times 8 - 3\times 4 = 16-12 = 4$

(b) $\begin{vmatrix} 5 & -4 \\ 3 & -2 \end{vmatrix} = 5\times(-2)-(-4)\times 3 = -10+12 = 2$

Consider the three equations in x, y, z :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Determinant of order three : $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Evaluation of third order determinant

Row-wise

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

Column-wise

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

Example

(a) $\begin{vmatrix} 1 & 2 & -3 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ -1 & 5 \end{vmatrix} - 2\begin{vmatrix} -2 & 4 \\ 2 & 5 \end{vmatrix} + (-3)\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix}$

$= 1\left(3\times5 - 4\times(-1)\right) - 2\left((-2)\times5 - 4\times2\right)$
$- 3\left((-2)\times(-1) - 3\times2\right)$

$= 1(15+4) - 2(-10-8) - 3(2-6)$

$= 19 + 36 + 12$

$= 67$

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(b) $\begin{vmatrix} 0 & -h & g \\ h & 0 & -f \\ -g & f & 0 \end{vmatrix}$

$= 0\begin{vmatrix} 0 & -f \\ f & 0 \end{vmatrix} - (-h)\begin{vmatrix} h & -f \\ -g & 0 \end{vmatrix} + g\begin{vmatrix} h & 0 \\ -g & f \end{vmatrix}$

$= 0 + h(0 - fg) + g(hf - 0)$

$= -hgf + hgf$

$= 0$

Minor & Cofactor

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Minor of } b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = a_2 c_3 - c_2 a_3.$$

$$\text{Minor of } c_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = a_1 b_3 - b_1 a_3$$

* Cofactor $= (-1)^{i+j} \times$ Minor of an element

$$\text{Cofactor of } b_1 = (-1)^{1+2} \times \text{Minor of } b_1$$

$$= (-1) \times \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$\text{Cofactor of } c_2 = (-1)^{2+3} \times \text{Minor of } c_2$$

$$= (-1) \times \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

Example:

Find the minor & cofactor of element 1 & 4 of the following determinant

$$\begin{vmatrix} 3 & -14 & 1 \\ 5 & 4 & -10 \\ -2 & 10 & -1 \end{vmatrix}$$

(a) Minor of 1 $= \begin{vmatrix} 5 & 4 \\ -2 & 10 \end{vmatrix} = 5 \times 10 - 4 \times (-2)$

$= 50 + 8 = 58$

Cofactor of 1 $= (-1)^{i+j} \times$ Minor of 1

$= (-1)^{1+3} \times 58 = (-1$

$= 58$

(b) Minor of 4 $= \begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} = 3 \times (-1) - 1 \times (-2)$

$= -3 + 2 = -1$

Cofactor of 4 $= (-1)^{2+2} \times$ Minor of 4

$= 1 \times (-1) = -1$ //

Properties of Determinant:

① If every element of a row (or column) of a determinant is zero, then the determinant vanishes or will be zero.

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

② If rows & columns are interchanged, then the value of a determinant remains unaltered.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Then $\Delta = \Delta'$

③ If two rows (or columns) of a determinant are interchanged then the value of the new determinant becomes (-1) time the value of original determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta = -\Delta'$$

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(4) If two rows (or columns) of a determinant are identical (or same), then the value of the determinant is zero.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(5) If all the elements of one row (or column) of a determinant be multiplied by the same constant number k (say), then the determinant itself is multiplied by that constant k.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} k a_1 & b_1 & c_1 \\ k a_2 & b_2 & c_2 \\ k a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k\Delta.$$

(6) If every element of a row (or column) of a determinant is the sum of two terms then the determinant can be expressed as the sum of two determinants:

$$\Delta = \begin{vmatrix} a_1+m_1 & b_1 & c_1 \\ a_2+m_2 & b_2 & c_2 \\ a_3+m_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}$$

① Prove that : $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Sol^n LHS $= \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$

$= \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ $R \leftrightarrow C$ by using property ②

$= (-1)\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$ $C_1 \leftrightarrow C_2$ by property ③

$= (-1)(-1)\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ $R_1 \leftrightarrow R_2$ by property ③

$$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = RHS$$

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(2) Prove That : $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$

Sol^n: LHS $= \begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$

$$= \begin{vmatrix} 1+a+b+c & b & c \\ 1+a+b+c & 1+b & c \\ 1+a+b+c & b & 1+c \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_2$$

$$= (1+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix} \text{, by property (5).}$$

$$= (1+a+b+c) \begin{vmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & b & 1+c \end{vmatrix} \quad \begin{matrix} R_1 \to R_2 - R_1 \\ R_2 \to R_3 - R_2 \end{matrix}$$

$$= (1+a+b+c) \left[0 - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1+c \end{vmatrix} + 0 \right]$$

$$= (1+a+b+c) \left[-1 (0-1) \right]$$

$$= (1+a+b+c)$$

$$= RHS.$$

(1) $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1 a_2 a_3 \left[1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right]$

Sol:ⁿ LHS $= \begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix}$

$= a_1 a_2 a_3 \begin{vmatrix} \frac{1+a_1}{a_1} & \frac{1}{a_1} & \frac{1}{a_1} \\ \frac{1}{a_2} & \frac{1+a_2}{a_2} & \frac{1}{a_2} \\ \frac{1}{a_3} & \frac{1}{a_3} & \frac{1+a_3}{a_3} \end{vmatrix}$

$= a_1 a_2 a_3 \begin{vmatrix} \frac{1}{a_1}+1 & \frac{1}{a_1} & \frac{1}{a_1} \\ \frac{1}{a_2} & \frac{1}{a_2}+1 & \frac{1}{a_2} \\ \frac{1}{a_3} & \frac{1}{a_3} & \frac{1}{a_3}+1 \end{vmatrix}$

$= a_1 a_2 a_3 \begin{vmatrix} 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} & 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} & 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} \\ \frac{1}{a_2} & \frac{1}{a_2}+1 & \frac{1}{a_2} \\ \frac{1}{a_3} & \frac{1}{a_3} & \frac{1}{a_3}+1 \end{vmatrix}$ $R_1 \to R_1+R_2+R_3$

$= a_1 a_2 a_3 \left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{a_2} & \frac{1}{a_2}+1 & \frac{1}{a_2} \\ \frac{1}{a_3} & \frac{1}{a_3} & \frac{1}{a_3}+1 \end{vmatrix}$

$$= a_1 a_2 a_3 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{a_2} & \frac{1}{a_2}+1 & \frac{1}{a_2} \\ \frac{1}{a_3} & \frac{1}{a_3} & \frac{1}{a_3}+1 \end{vmatrix}$$

$$= a_1 a_2 a_3 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & \frac{1}{a_2} \\ 0 & 1 & \frac{1}{a_3}+1 \end{vmatrix}$$

$C_1 \to C_2 - C_1$

$C_2 \to C_3 - C_2$

$$= a_1 a_2 a_3 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \left[0 - 0 + 1 \cdot \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}\right].$$

$$= a_1 a_2 a_3 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \left[1(1 - 0)\right]$$

$$= a_1 a_2 a_3 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) = \text{RHS}$$

(2) Show that $\begin{vmatrix} x+y & z & z-x \\ y+z & x & x-y \\ z+x & y & y-z \end{vmatrix} = x^3+y^3+z^3-3xyz$

Sol^n:

$$\begin{vmatrix} x+y & z & z-x \\ y+z & x & x-y \\ z+x & y & y-z \end{vmatrix}$$

$\cancel{z}-x-\cancel{z}$

$$= \begin{vmatrix} x+y+z & z & -x \\ x+y+z & x & -y \\ x+y+z & y & -z \end{vmatrix}$$

$C_1 \to C_1 + C_2$

$C_3 \to C_3 - C_2$

$$= (x+y+z) \begin{vmatrix} 1 & z & -x \\ 1 & x & -y \\ 1 & y & -z \end{vmatrix}$$

$\boxed{\begin{aligned} &-y-(-z) \\ &= -y+z \end{aligned}}$

$$= (x+y+z) \begin{vmatrix} 0 & z-y & z-x \\ 0 & x-y & z-y \\ 1 & y & -z \end{vmatrix}$$

$R_1 \to R_1 - R_3$

$R_2 \to R_2 - R_3$

$$= (x+y+z)\left[0 - 0 + 1 \begin{vmatrix} z-y & z-x \\ x-y & z-y \end{vmatrix}\right]$$

$$= (x+y+z)\left[(z-y)(z-y) - (z-x)(x-y)\right]$$

$$= (x+y+z)\left[(z-y)^2 - (zx - zy - x^2 + xy)\right]$$

$$= (x+y+z)\left[z^2 - 2yz + y^2 - zx + zy + x^2 - xy\right]$$

$$= (x+y+z)(x^2+y^2+z^2 - xy - yz - zx)$$

$$= x^3+y^3+z^3 - 3xyz = \text{RHS}$$

Cramer's Rule

Two linear equations in two unknowns:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

Then $x = \dfrac{\Delta_x}{\Delta}$, $y = \dfrac{\Delta_y}{\Delta}$, where $\Delta \neq 0$.

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Three linear equations in ~~two~~ three unknowns:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Then $x = \dfrac{\Delta_x}{\Delta}$, $y = \dfrac{\Delta_y}{\Delta}$, $z = \dfrac{\Delta_z}{\Delta}$, where $\Delta \neq 0$.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Ex (1): Solve by Cramer's rule:

$$3x - 5y = 7$$
$$4x + y = 17$$

Sol.^{n}

$$\Delta = \begin{vmatrix} 3 & -5 \\ 4 & 1 \end{vmatrix} = 3 \times 1 - (-5) \times 4 = 23 \neq 0$$

$$\Delta_x = \begin{vmatrix} 7 & -5 \\ 17 & 1 \end{vmatrix} = 7 \times 1 - (-5) \times 17 = 92$$

$$\Delta_y = \begin{vmatrix} 3 & 7 \\ 4 & 17 \end{vmatrix} = 3 \times 17 - 7 \times 4 = 23$$

$$x = \frac{\Delta_x}{\Delta} = \frac{92}{23} = 4$$

$$y = \frac{\Delta_y}{\Delta} = \frac{23}{23} = 1$$

Ex② Solve by Cramer's rule : $x + y + z = 3$

$2x + 3y + 4z = 9$

$x + 2y - 4z = -1$

Solⁿ

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -7 \neq 0.$$

$$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 4 \\ -1 & 2 & -4 \end{vmatrix} = 3\begin{vmatrix} 3 & 4 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 9 & 4 \\ -1 & -4 \end{vmatrix} + 1\begin{vmatrix} 9 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= -7$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 9 & 4 \\ 1 & -1 & -4 \end{vmatrix} = 1\begin{vmatrix} 9 & 4 \\ -1 & -4 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} 2 & 9 \\ 1 & -1 \end{vmatrix}$$

$$= -7$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 1 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 9 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -7$$

$\therefore x = \dfrac{\Delta_x}{\Delta} = \dfrac{-7}{-7} = 1$, $\quad y = \dfrac{\Delta_y}{\Delta} = \dfrac{-7}{-7} = 1$, $\quad z = \dfrac{\Delta_z}{\Delta} = \dfrac{-7}{-7} = 1$

Ex ③ Solve by Cramer's rule:

$$x + 2z = 7$$
$$3x + y = 5$$
$$2y - 3z = -5$$

Solⁿ:

$$x + 0.y + 2z = 7$$
$$3x + y + 0.z = 5$$
$$0x + 2y - 3z = -5$$

$$\Delta = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & -3 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} - 0 + 2\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$
$$= 1(-3-0) + 2(6-0)$$
$$= -3 + 12$$
$$= 9 \neq 0$$

$$\Delta_x = \begin{vmatrix} 7 & 0 & 2 \\ 5 & 1 & 0 \\ -5 & 2 & -3 \end{vmatrix} = 7\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} - 0 + 2\begin{vmatrix} 5 & 1 \\ -5 & 2 \end{vmatrix}$$
$$= 7(-3-0) + 2(10+5)$$
$$= -21 + 2 \times 15$$
$$= -21 + 30$$
$$= 9$$

$$\Delta_y = \begin{vmatrix} 1 & 7 & 2 \\ 3 & 5 & 0 \\ 0 & -5 & -3 \end{vmatrix} = 1\begin{vmatrix} 5 & 0 \\ -5 & -3 \end{vmatrix} - 3\begin{vmatrix} 7 & 2 \\ -5 & -3 \end{vmatrix} + 0$$
$$= 1(-15+0) - 3(-21+10)$$
$$= -15 + 3 \times 11$$
$$= -15 + 33$$
$$= 18$$

$$\Delta_z = \begin{vmatrix} 1 & 0 & 7 \\ 3 & 1 & 5 \\ 0 & 2 & -5 \end{vmatrix} = 1\begin{vmatrix} 1 & 5 \\ 2 & -5 \end{vmatrix} - 0 + 7\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$
$$= 1(-5-10) + 7(6-0)$$
$$= -15 + 42$$
$$= 27$$

$$\therefore \quad x = \frac{\Delta_x}{\Delta} = \frac{9}{9} \quad , \quad y = \frac{\Delta_y}{\Delta} = \frac{18}{9} = 2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{27}{9} = 3$$