Recurrence Relations

A recurrence relation is a functional relation between the independent variable x, dependent variable f(x) and the differences of various order of f (x). A recurrence relation is also called a difference equation, and we will use these two terms interchangeably.

Example1: The equation f (x + 3h) + 3f (x + 2h) + 6f (x + h) + 9f (x) = 0 is a recurrence relation.

It can also be written as

$$a_{r+3} + 3a_{r+2} + 6a_{r+1} + 9a_r = 0$$

$$y_{k+3} + 3y_{k+2} + 6y_{k+1} + 9y_k = 0$$

Example2: The Fibonacci sequence is defined by the recurrence relation $a_r = a_{r-2} + a_{r-1}$, $r \geq 2$,with the initial conditions $a_0=1$ and $a_1=1$.

Order of the Recurrence Relation:

The order of the recurrence relation or difference equation is defined to be the difference between the highest and lowest subscripts of f(x) or $a_r=y_k$.

Example1: The equation $13a_r+20a_{r-1}=0$ is a first order recurrence relation.

Example2: The equation 8f (x) + 4f (x + 1) + 8f (x+2) = k (x)

Degree of the Difference Equation:

The degree of a difference equation is defined to be the highest power of f (x) or $a_r=y_k$

Example1: The equation $y^3_{k+3}+2y^2_{k+2}+2y_{k+1}=0$ has the degree 3, as the highest power of y_k is 3.

Example2: The equation $a^4_r+3a^3_{r-1}+6a^2_{r-2}+4a_{r-3} =0$ has the degree 4, as the highest power of a_r is 4.

Example3: The equation $y_{k+3} +2y_{k+2} +4y_{k+1}+2y_k= k(x)$ has the degree 1, because the highest power of y_k is 1 and its order is 3.

Example4: The equation f (x+2h) - 4f(x+h) +2f(x) = 0 has the degree1 and its order is 2.

Linear Recurrence Relations with Constant Coefficients

A Recurrence Relations is called linear if its degree is one.

The general form of linear recurrence relation with constant coefficient is

$C_0\ y_{n+r}+C_1\ y_{n+r-1}+C_2\ y_{n+r-2}+\cdots+C_r\ y_n=R\ (n)$

Where $C_0,C_1,C_2......C_n$ are constant and R (n) is same function of independent variable n.

A solution of a recurrence relation in any function which satisfies the given equation.

Linear Homogeneous Recurrence Relations with Constant Coefficients:

The equation is said to be linear homogeneous difference equation if and only if R (n) = 0 and it will be of order n.

The equation is said to be linear non-homogeneous difference equation if R (n) ≠ 0.

Example1: The equation $a_{r+3}+6a_{r+2}+12a_{r+1}+8a_r=0$ is a linear non-homogeneous equation of order 3.

Example2: The equation $a_{r+2}-4a_{r+1}+4a_r= 3r + 2^r$ is a linear non-homogeneous equation of order 2.

A linear homogeneous difference equation with constant coefficients is given by

$C_0\ y_n+C_1\ y_{n-1}+C_2\ y_{n-2}+\cdots......+C_r\ y_{n-r}=0$ .... equation (i)

Where $\mathbf{C_0,C_1,C_2.....C_n}$ are constants.

The solution of the equation (i) is of the form $A\propto_1^K$, where $\propto_1$ is the characteristics root and A is constant.

Substitute the values of $A \propto^K$ for y_n in equation (1), we have

$C_0\ A\propto^K+C_1\ A\propto^{K-1}+C_2\ A\propto^{K-2}+\cdots....+C_r\ A\propto^{K-r}=0$.......equation (ii)

After simplifying equation (ii), we have

$C_0 \propto^r+C_1 \propto^{r-1}+C_2 \propto^{r-2}+\cdots C_r=0$..........equation (iii)

The equation (iii) is called the characteristics equation of the difference equation.

If $\propto_1$ is one of the roots of the characteristics equation, then $A\propto_1^K$ is a homogeneous solution to the difference equation.

To find the solution of the linear homogeneous difference equations, we have the four cases that are discussed as follows:

Case1: If the characteristic equation has n distinct real roots $\propto_1, \propto_2, \propto_3,.......\propto_n$.

Thus, $\alpha_1^K, \alpha_2^K,........ \alpha_n^K$ are all solutions of equation (i).

Also, we have $A_1 \propto_1^K, A_2 \propto_1^K,.........A_n \propto_n^K$ are all solutions of equation (i). The sums of solutions are also solutions.

Hence, the homogeneous solutions of the difference equation are

$$y_k = A_1 \propto_1^K, A_2 \propto_2^K, \ldots\ldots\ldots A_n \propto_n^K.$$

Case2: If the characteristics equation has repeated real roots.

If $\propto_1 = \propto_2$, then $(A_1 + A_2 K) \propto_1^K$ is also a solution.

If $\propto_1 = \propto_2 = \propto_3$ then $(A_1 + A_2 K + A_3 K^2) \propto_1^K$ is also a solution.

Similarly, if root $\propto_1$ is repeated n times, then.

$$(A_1 + A_2 K + A_3 K^2 + \ldots\ldots + A_n K_{n-1}) \propto_1^K$$

The solution to the homogeneous equation.

Case3: If the characteristics equation has one imaginary root.

If $\alpha+i\beta$ is the root of the characteristics equation, then $\alpha-i\beta$ is also the root, where α and β are real.

Thus, $(\alpha+i\beta)^K$ and $(\alpha-i\beta)^K$ are solutions of the equations. This implies

$$(\alpha+i\beta)^K A_1 + \alpha-i\beta)^K A_2$$

Is also a solution to the characteristics equation, where A_1 and A_2 are constants which are to be determined.

Case4: If the characteristics equation has repeated imaginary roots.

When the characteristics equation has repeated imaginary roots,

$$(C_1 + C_2 k) (\alpha+i\beta)^K + (C_3 + C_4 K)(\alpha-i\beta)^K$$

Is the solution to the homogeneous equation.

Example1: Solve the difference equation $a_{r-3ar-1+2ar-2=0.}$

Solution: The characteristics equation is given by

$$s^2-3s+2=0 \text{ or } (s-1)(s-2)=0$$
$$\Rightarrow s = 1, 2$$

Therefore, the homogeneous solution of the equation is given by

$$a_r = C^1{}_r + C^2.2^r.$$

Example2: Solve the difference equation $9y_{K+2}-6y_{K+1}+y_K=0$.

Solution: The characteristics equation is

$9s^2-6s+1=0$ or $(3s-1)^2=0$

$\Rightarrow s = \frac{1}{3}$ and $\frac{1}{3}$

Therefore, the homogeneous solution of the equation is given by

$y_K=(C_1+C_2\ k).\left(\frac{1}{3}\right)^K$

Example3: Solve the difference equation $y_K-y_{K-1}-y_{K-2}=0$.

Solution: The characteristics equation is $s^2-s-1=0$

$s=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$

Therefore, the homogeneous solution of the equation is

$$y_K = C_1\left[\frac{1+\sqrt{5}}{2}\right]^K+C_2\left[\frac{1-\sqrt{5}}{2}\right]^K$$

Example4: Solve the difference equation $y_{K+4}+4y_{K+3}+8y_{K+2}+8y_{K+1}+4y_K=0$.

Solution: The characteristics equation is $s^4+4s^3+8s^2+8s+4=0$

$(s^2+2s+2)\ (s^2+2s+2)=0$

$s = -1\pm i, -1\pm i$

Therefore, the homogeneous solution of the equation is given by

$y_K=(C_1+C_2\ K)(-1+i)^K+(C_3\ +C_4\ K)(-1-i)^K$

Particular Solution

(a) Homogeneous Linear Difference Equations and Particular Solution:

We can find the particular solution of the difference equation when the equation is of homogeneous linear type by putting the values of the initial conditions in the homogeneous solutions.

Example1: Solve the difference equation $2a_r-5a_{r-1}+2a_{r-2}=0$ and find particular solutions such that $a_0=0$ and $a_1=1$.

Solution: The characteristics equation is $2s^2-5s+2=0$

$(2s-1)(s-2)=0$

$s = \frac{1}{2}$ and 2.

Therefore, the homogeneous solution of the equation is given by

$a_{r(h)}= C_1\left(\frac{1}{2}\right)^r+C_2\ .2^r$..........equation (i)

Putting r=0 and r=1 in equation (i), we get

$a_0=C_1+C_2=0$..........equation (a)

$a_1= \frac{1}{2} C_1+2C_2=1$..........equation.(b)

Solving eq (a) and (b), we have

$C_1=-\frac{2}{3}$ and $C_2=\frac{2}{3}$

Hence, the particular solution is

$$a_{r(P)} = -\frac{2}{3}.\left(\frac{1}{2}\right)^r+\frac{2}{3}.(2)^r$$

Example2: Solve the difference equation $a_r-4a_{r-1}+4a_{r-2}=0$ and find particular solutions such that $a_0=0$ and $a_1=6$.

Solution: The characteristics equation is

$s^2-4s+4=0$ or $(s-2)^2=0$ $\quad$ s = 2, 2

Therefore, the homogeneous solution of the equation is given by

$a_{r(n)}=(C_1+C_2\ r).2^r$............. equation (i)

Putting r = 0 and r = 1 in equation (i), we get

$a_0=(C_1+0).2^0 = 1$ $\quad$ $\therefore C_1=1$

$a_1=(C_1+C_2).2=6$ $\quad$ $\therefore C_1+C_2=3 \Rightarrow C_2=2$

Hence, the particular solution is

$a_{r(P)}=(1+2r).2^r$.

Example3: Solve the difference equation $9a_r-6a_{r-1}+a_{r-2}=0$ satisfying the conditions $a_0=0$ and $a_1=2$.

Solution: The characteristics equation is

$9s^2-6s+1=0$ or $(3s-1)^2=0$

$s = \frac{1}{3}, \frac{1}{3}$

Therefore, the homogeneous solution of the equation is given by

$a_{r(h)}=(C_1+C_2\ r).\left(\frac{1}{3}\right)^r$ ....equation (i)

Putting r = 0 and r = 1 in equation (i), we get

$a_0=C\_1=0$

$a_1= (C_1+C_2).\frac{1}{3}=2.$ $\quad$ $\therefore C_1+C_2=6 \Rightarrow C_2=6$

Hence, the particular solution is

$a_{r(P)}=6r.\left(\frac{1}{3}\right)^r$.

(b) Non-Homogeneous Linear Difference Equations and Particular Solution:

There are two methods to find the particular solution of a non-homogeneous linear difference equation. These are as follows:

1. Undetermined coefficients method
2. E and Δ operator method.

1. Undetermined Coefficients Method: This method is used to find a particular solution of non-homogeneous linear difference equations, whose R.H.S term R (n) consist of terms of special forms.

In this method, firstly we assume the general form of the particular solutions according to the type of R (n) containing some unknown constant coefficients, which have to be determined. Then according to the difference equation, we will determine the exact solution.

The general form of a particular solution to be assumed for the special forms of R (n), to find the exact solution is shown in the table.

Form of R (n)	General form to be assumed
Z, here z is constant	A
Z^r, here z is constant	Z^r
P (r), a polynomial of degree n	$A_0 r^n+A_1 r^{n-1}+\cdots..A_n$
Z^r. P (r), here P(r) is a polynomial of the nth degree in r. Z is a constant.	$[A_0 r^n+A_1 r^{n-1}+\cdots..A_n].Z^r$

Example1: Find the particular solution of the difference equation $a_{r+2}-3a_{r+1}+2a_r=Z^r$ ....equation (i)

Where Z is some constant.

Solution: The general form of solution is = A. Z^r

Now putting this solution on L.H.S of equation (i), we get
$= A Z^{r+2}-3AZ^{r+1}+2AZ^r=(Z^2-3Z+2) A Z^r$.........equation (ii)

Equating equation (ii) with R.H.S of equation (i), we get
$(Z^2-3Z+2)A=1$

$$A = \frac{1}{(Z^2-3Z+2)} = \frac{1}{(Z-1)(Z-2)} \quad (Z\neq 1, Z\neq 2)$$

Therefore, the particular solution is $\frac{Z^r}{(Z-1)(Z-2)}$

Example2: Find the particular solution of the difference equation $a_{r+2}-5a_{r+1}+6a_r=5^r$ ....equation (i)

Solution: Let us assume the general form of the solution= $A. 5^r$.

Now to find the value of A, put this solution on L.H.S of the equation (i), then this becomes

$= A. 5^{r+2}-5.A5^{r+1}+6.A5^r$
$= 25A. 5^r-25A.5^r+6A.5^r$
$= 6A.5^r$ ....equation (ii)

Equating equation (ii) to R.H.S of equation (i), we get

$A = \frac{1}{6}$

Therefore, the particular solution of the difference equation is $= \frac{1}{6}.5^r$.

Example3: Find the particular solution of the difference equation $a_{r+2}+a_{r+1}+a_r=r.2^r$..........equation (i)

Solution: Let us assume the general form of the solution = (A_0+A_1r). 2^r

Now, put these solutions in the L.H.S of the equation (i), we get

$= 2^{r+2} [A_0+A_1 (r+2)]+2^{r+1} [A_0+A_1 (r+1)]+2^r (A_0+A_1 r)$
$= 4. 2^r (A_0+A_1 r+2A_1)+2.2^r (A_0+A_1 r+A_1)+2^r (A_0+A_1 r)$
$= r. 2^r (7A_1)+2^r (7A_0+10A_1)$............equation (ii)

Equating equation (ii) with R.H.S of equation (i), we get

$7A_1=1$ $\quad\therefore A_1= \frac{1}{7}$

$7A_0+10A_1=0$ $\quad\therefore A_0= \frac{-10}{49}$

Therefore, the particular solution is $2^r\left(\frac{-10}{49}+\frac{1}{7}r\right)$

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2. E and Δ operator Method:

Definition of Operator E: The operator of E on f(x) means that give an increment to the value of x in the function. The operation of E is, put (x+h) in the function wherever there is x. Here h is increment quantity. So Ef(x) = f(x+h)

Here, E is operated on f(x), therefore, E is a symbol known as shift operator.

Definition of OperatorΔ: The operation Δ is an operation of two steps.

Firstly, x in the function is incremented by a constant and then former is subtracted from the later i.e.,

$\Delta f(x)=f(x+h)-f(x)$

Theorem1: Prove that $E \cong 1+\Delta$.

Proof: The operation of Δ on f(x) is of two steps. First, increment the value of x in the function. So, whenever, there is x in f(x) put x+h (here h is constant increment), which means operation of E on f(x) i.e.,

$f (x+h)=Ef(x)$.

Second, subtract the original function from the value obtained in the first step, hence

$$\Delta f(x)=Ef(x)-1f(x)=(E-1)f(x)$$

So, the operation of Δ on f(x) is equivalent to the operation of (E-1) on f(x).

Therefore, we have

$$E \cong 1+\Delta.$$

Theorem2: Show that $E^n f(x)=f(x+nh)$.

Proof: We know that $E f(x) = f(x+h)$

Now $E^n f(x)=E.E.E.E.........n \text{ times } f(x)$

$= E^{n-1} [E f(x)] = E^{n-1} f(x+h)$

$= E^{n-2} [E f(x+h)] = E^{n-2} f(x+2h)$

....

....

$= E f[x+ (n-1) h] = f(x+nh)$.

Theorem3: Show that $E Cf(x) = CE f(x)$

Proof: We know that $E C f(x) = C f(x+h) = CE f(x+h)$. Hence Proved.

There is no effect of the operation of E on any constant. Therefore, the operation of E on any constant will be equal to constant itself.

By E and Δ operator method, we will find the solution of

$C_0 y_{n+r}+C_1 y_{n+r-1}+C_2 y_{n+r-2}+\cdots+C_n y_n=R(n)$..............equation (i)

Equation (i) can be written as

$C_0 E^r y_n+C_1 E^{r-1} y_n+C_2 E^{r-2} y_n+\cdots+C_n y_n=R(n)$

$(C_0 E^r+C_1 E^{r-1}+C_2 E^{r-2}+\cdots+C_n) y_n=R(n)$

Putting $C_0 E^r+C_1 E^{r-1}+C_2 E^{r-2}+\cdots+C_n=P(E)$

So $P(E) y_n=R(n)$

$\therefore \quad y_n=\frac{R(n)}{P(E)}$................equation (ii)

To find the particular solution of (ii) for different forms of R (n), we have the following cases.

Case1: When R (n) is some constant A.

We know that, the operation of E on any constant will be equal to the constant itself i.e.,

$EA=A$

Therefore, $P(E) A = (C_0 E^r+C_1 E^{r-1}+C_2 E^{r-2}+\cdots+C_n)A$

$= (C_0+C_1+C_2+\cdots+C_n)A$

$= P(1) A$

$$\frac{A}{P(E)} = \frac{1}{P(1)}.A$$

Therefore, using equation (ii), the particular solution of (i) is

$$y_n= \frac{A}{P(1)}, P(1)\neq 0$$

P (1) is obtained by putting E = 1 in P (E).

Case2: When R (n) is of the form $A. Z^n$, where A and Z are constants

We have, $P(E)(A. Z^n)=\{C_0 E^r+C_1 E^{r-1}+\cdots+ C_n\}(A.Z^n)$
$=A\{C_0 Z^{r+n}+C_1 Z^{r+n-1}+\cdots+C_n Z^n\}$
$= A\{C_0 Z^r+C_1 Z^{r-1}+\cdots+C_n \}. Z^n$
$=AP(Z).Z^n$

To get, P (Z) put E=Z in P (E)

Therefore, $\frac{A.Z^n}{P(E)}=\frac{A.Z^n}{P(Z)}$, provided $P(Z) \neq 0$

Thus, $y_n=\frac{A.Z^n}{P(Z)}$, $P(Z) \neq 0$

If A = 1, then $y_n=\frac{Z^n}{P(Z)}$

When P (Z) = 0 then for equation

(i) $(E-Z) y_n= A. Z^n$

For this, the particular solution becomes $A. \frac{1}{(E-Z)} Z^n=A. n Z^n-1$

(ii) $(E-Z)^2 y_n= A. Z^n$

For this, the particular solution becomes $A. \frac{1}{(E-Z)^2}. Z^n=\frac{A.n(n-1)}{2!}. Z^{n-2}$

(iii) $(E-Z)^3 y_n= A. Z^n$

For this, the particular solution becomes $A. \frac{1}{(E-Z)^3}. Z^n=\frac{A.n(n-1)(n-2)}{3!}. Z^{n-3}$ and so on.

Case3: When R (n) be a polynomial of degree m is n.

We know that $E\cong 1+\Delta$
So, $P(E) =P(1+\Delta)$

$$\frac{1}{P(E)} = \frac{1}{P(1+\Delta)}$$

Which can be expanded in ascending power of Δ as far as upto Δ^m

$\Rightarrow \frac{1}{P(E)} = \frac{1}{P(1+\Delta)} =(b_0+b_1 \Delta+b_2 \Delta+\cdots.+b_m \Delta^m+\cdots)$

$\Rightarrow \frac{1}{P(E)}.R(n)=(b_0+b_1 \Delta+b_2 \Delta+\cdots.+b_m \Delta^m+\cdots).R(n)$
$= b_0 R(n)+b1 \Delta R(n)+\cdots+bm \Delta^m R(n)$

All other higher terms will be zero because R (n) is a polynomial of degree m.

Thus, the particular solution of equation (i), in this case will be

$$y_n=b_0 R(n)+b_1 \Delta R(n)+\cdots.+b_m \Delta^m R(n).$$

Case4: When R (n) is of the form $R(n).Z^n$,where R(n) is a polynomial of degree m and Z is some constant

We have $E^r[Z^n R(n)]=Z^{r+n} R(n+r)=Z^r.Z^n.E^r.R(n)=Z^n (ZE)^r R(n)$

Similarly, we have

$$\frac{1}{P(E)} [Z^n R(n)]=Z^n \frac{1}{P(ZE)} .(R(n))= Z^n [P(Z+Z\Delta)]^{-1}.R(n)$$

Thus, the particular solution of equation (i), in this case will be

$$y_n=Z^n [P(Z+Z\Delta)]^{-1}.R(n)$$

Example1: Find the particular solution of the difference equation

$$2a_{r+1}-a_r=12.$$

Solution: The above equation can be written as

$$(2E-1) a_r=12$$

The particular solution is given by

$$a_r= \frac{1}{(2E-1)}.12$$

Put E=1, in the equation. The particular solution is $a_r=12$

Example2: Find the particular solution of the difference equation $a_r-4a_{r-1}+4a_{r-2}=2^r$.

Solution: The above equation can be written as

$$(E^2-4E+4) a_r=2^r$$

Therefore, $P(E) = E^2-4E+4 = (E-2)^2$

Thus, the particular solution is given by

$$a_r = \frac{1}{(E-2)^2}.2^r = \frac{r(r-1)}{2}.2^{r-2}$$

$$a_r = r(r-1).2^{r-3}$$

Total Solution

The total solution or the general solution of a non-homogeneous linear difference equation with constant coefficients is the sum of the homogeneous solution and a particular solution. If no initial conditions are given, obtain n linear equations in n unknowns and solve them, if possible to get total solutions.

If $y_{(h)}$ denotes the homogeneous solution of the recurrence relation and $y_{(p)}$ indicates the particular solution of the recurrence relation then, the total solution or the general solution y of the recurrence relation is given by

$$y =y_{(h)}+y_{(p)}.$$

Example: Solve the difference equation

$a_r-4a_{r-1}+4a_{r-2}=3r+2^r$...........equation (i)

Solution: The homogeneous solution of this equation is obtained by putting R.H.S equal to zero i.e.,

$a_r-4a_{r-1}+4a_{r-2}=0$

The homogeneous solution is $a_{r(h)}= (C_1+C_2\ r).2^r$

The equation (i) can be written as $(E^2-4E+4)\ a_r=3r+2^r$

The particular solution is given as

$$a_{r(p)} = \frac{1}{(E^2-4E+4)}.(3r+2^r) = \frac{1}{(E-2)^2}.(3r) + \frac{1}{(E-2)^2}.2^r$$

$$= 3.\frac{1}{(1-\Delta)^2}(r) + \frac{r(r-1)}{2!}.2^{r-2} = 3(1-\Delta)^{-2}(r) + \frac{r(r-1)}{2!}.2^{r-2}$$

$$=3\ (1+2\Delta).[r]+\frac{r(r-1)}{2!}.2^{r-2}=3(r+2)+r(r-1).\ 2^{r-3}$$

$$a_{r(p)} = 3(r+2) + r(r-1).2^{r-3}$$

Therefore, the total solution is $a_r = (C_1 + C_2 r).2^r + 3(r+2) + r(r-1).2^{r-3}$

Generating Functions

Generating function is a method to solve the recurrence relations.

Let us consider, the sequence a_0, a_1, a_2....a_r of real numbers. For some interval of real numbers containing zero values at t is given, the function G(t) is defined by the series

$G(t)= a_0, a_1t+a_2\ t^2+\cdots+a_r\ t_r+$............equation (i)

This function G(t) is called the generating function of the sequence a_r.

Now, for the constant sequence 1, 1, 1, 1.....the generating function is

$$G(t) = \frac{1}{(1-t)}$$

It can be expressed as

$G(t) =(1-t)^{-1}=1+t+t^2 +t^3+t^4+\cdots$[By binomial expansion]

Comparing, this with equation (i), we get

$a_0=1,a_1=1,a_2=1$ and so on.

For, the constant sequence 1,2,3,4,5,..the generating function is

$G(t) = \frac{1}{(1-t)^2}$ because it can be expressed as

$G(t) =(1-t)^{-2}=1+2t+3t^2 +4t^3+\cdots+(r+1)\ t^r$

Comparing, this with equation (i), we get
$a_0=1, a_1=2, a_2=3, a_3=4$ and so on.

The generating function of Z^r,($Z\neq0$ and Z is a constant)is given by

$G(t)= 1+Zt+Z^2 t^2+Z^3 t^3+\cdots+Z^r t^r$

$G(t)=\frac{1}{(1-Zt)}$ [Assume $|Zt|<1$]

So, $G(t)=\frac{1}{(1-Zt)}$ generates $Z^r, Z\neq0$

Also,If $a^{(1)}_r$ has the generating function $G_1(t)$ and $a^{(2)}_r$ has the generating function $G_2(t)$, then $\lambda_1 a^{(1)}_r+\lambda_2 a^{(2)}_r$ has the generating function $\lambda_1 G_1(t)+ \lambda_2 G_2(t)$. Here λ_1 and λ_2 are constants.

Application Areas:

Generating functions can be used for the following purposes -

- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

Example: Solve the recurrence relation $a_{r+2}-3a_{r+1}+2a_r=0$

By the method of generating functions with the initial conditions $a_0=2$ and $a_1=3$.

Solution: Let us assume that

$$G(t)=\sum_0^\infty a_r t^r$$

Multiply equation (i) by t^r and summing from r = 0 to ∞, we have

$$\sum_0^\infty a_{r+2} t^r - 3\sum_0^\infty a_{r+1} t^r + 2\sum_0^\infty a_r t^r=0$$

$(a_2+a_3 t+a_4 t^2+\cdots)-3(a_1+a_2 t+a_3 t^2+\cdots)+2(a_0+a_1 t+a_2 t^2+\cdots)=0$
[$\therefore G(t)=a_0+a_1 t+a_2 t^2+\cdots$]

$$\therefore \quad \frac{G(t)-a_0-a_1 t}{t^2} - 3\left(\frac{G(t)-a_0}{t}\right) +2G(t)=0 \text{............equation (ii)}$$

Now, put $a_0=2$ and $a_1=3$ in equation (ii) and solving, we get

$$G(t) = \frac{2-3t}{1-3t+2t^2} \text{ or } G(t) = \frac{2-3t}{(1-t)(1-2t)}$$

Now, Let $\frac{2-3t}{(1-t)(1-2t)} = \frac{A}{1-t} + \frac{B}{1-2t}$

i.e., $2-3t = A(1-2t)+B(1-t)$........................equation (iii)

Put t=1 on both sides of equation (iii) to find A. Hence

$-1 = -A \quad \therefore A = 1$

Put $t = \frac{1}{2}$ on both sides of equation (iii) to find B. Hence

$\frac{1}{2} = \frac{1}{2} B \quad \therefore B = 1$

Thus $G(t) = \frac{1}{1-t} + \frac{1}{1-2t}$. Hence, $a_r = 1 + 2^r$.