Differential Equation

A differential equation is one which connects an independent variable (x), a dependent variable (y) which is a function $y = f(x)$ and its derivatives (differential co-efficients) y', y'' . . . . $y^{(n)}$.

Let us explain the concept of differential equation considering a function $y = f(x)$ whose derivative is given by

$$\frac{dy}{dx} = f'(x). \quad \text{——— (1)}$$

we can find out the original function taking integral of $dy = f'(x)\, dx$.

Thus,

$$y = \int f'(x)\, dx \quad \text{——— (2)}$$

which means we have solved the equation $\frac{dy}{dx} = f'(x)$ and it can be termed as differential ~~equa~~ equation.

Thus, differential equations are equations that involve derivatives.

*FIRST ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT TERM & CONSTAN CO-EFFICIENT :

In a differential equation when the derivative $\frac{dy}{dx}$ and the dependent variable y appear in first degree and no product of the form $y\left(\frac{dy}{dx}\right)$ appears, then the equation will be called linear. So, the first order linear differential equation will take the general form as

$$\frac{dy}{dx} + u(x)\,y = v(x) \qquad \text{——— (3)}$$

Where u and v like y, are two f^n of x. When u and v are constants, then the first order linear differential equation (3) reduces to

$$\frac{dy}{dx} + ay = b \qquad \text{——— (4)}$$

which we call a first order linear differential eqtn with constant co-efficient and constant term.

When the constant term on the right hand side of (4) is zero, the differential equation

$$\frac{dy}{dx} + ay = 0 \quad \text{--- (5)}$$

is said to be homogeneous. Otherwise eqtn (4) is a non-homogeneous Linear Diff. Eqtn.

Let us first find the solution of homogeneous diff. eqtn (5) as

$$\frac{dy}{dx} = -ay$$

or, $$\frac{1}{y}\frac{dy}{dx} = -a \quad \text{--- (6)}$$

Taking integration on both sides of (6) ~~and~~ will get the general solution as

$$\int \frac{1}{y}\frac{dy}{dx} = -\int a$$

or $$\int \frac{1}{y} dy = -\int a\, dx$$

or, $$\log(y) + c_1 = -ax + c_2$$

$$\log(y) = -ax + c \quad [\because c_1 + c_2 = c]$$

$$y(x) = e^{-ax+c}$$

or, $y(x) = A e^{-ax}$ $\left[\because e^{c} = A\right]$ → (7)

putting $x=0$, $y(0) = A$, the definite solution is

$$\boxed{y(x) = y(0)\, A e^{-ax}} \quad \text{—— (8)}$$

But when we have a non-homogeneous first order differential eqn (4)

$$\frac{dy}{dx} + ay = b$$

The solution of this eqn will consist of two parts — Complementary solution & particular solution. The sum of these two will give us the complete ~~solution~~ general solution. The complementary solution is nothing but the

Solution of the homogeneous version of (4), that is $\frac{dy}{dx} + ay = 0$,

the general solution of which is already shown to be

$$y_c = Ae^{-ax}$$

where, y_c = complementary solution.

Since the particular integral is any particular solution, we can consider it to be a constant, say $y = c$. When y is a constant, $\frac{dy}{dx} = 0$ in eqn (4) and so we will have

$$ay = b$$

or $y(=y_p) = \frac{b}{a}$, $a \neq 0$

So the complete solution is given by the sum of y_c and y_p. Thus, the general solution of complete eqn (4) is given by

$$y(x) = y_c + y_p$$

$$= Ae^{-ax} + \frac{b}{a} \quad \text{———} \quad (9)$$

Taking the initial condition $x = 0$

$$y(0) = Ae^{0} + \frac{b}{a}$$

$$\therefore A = \left[y(0) - \frac{b}{a}\right] \quad \text{--- (10)}$$

Substituting (10) in (9), we get the final form of the definite solution

$$y(x) = \left[y(0) - \frac{b}{a}\right]e^{-ax} + \frac{b}{a}$$

assuming $a \neq 0$.