Recurrence Relation of Binomial Distribution

The recurrence relation of the probabilities of binomial distribution is given by the following expression:

$$P(X = x+1) = \frac{n-x}{x+1} \times \frac{p}{q} P(x).$$

Proof: We know, the pmf of Binomial distribution is given by

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}, \quad x = 0, 1, 2, \dots, n. \longrightarrow (1)$$

Now, $P(x+1) = {}^{n}C_{x+1}\, p^{x+1} q^{n-(x+1)} \longrightarrow (2)$

Dividing eq^{n} (2) by (1)

$$\frac{P(x+1)}{P(x)} = \frac{{}^{n}C_{x+1}\, p^{x+1} q^{n-x-1}}{{}^{n}C_{x}\, p^{x} q^{n-x}}$$

$$= \frac{{}^{n}C_{x+1}}{{}^{n}C_{x}} \cdot p^{x+1-x} \cdot q^{n-x-1-n+x}$$

$$= \frac{\dfrac{n!}{(x+1)!\,(n-x-1)!}}{\dfrac{n!}{x!\,(n-x)!}} \quad \frac{p}{q}$$

$$= \frac{n!}{(x+1)!\,(n-x-1)!} \times \frac{x!\,(n-x)!}{n!} \quad \frac{p}{q}$$

$$= \frac{x!\,(n-x)\,\cancel{(n-x-1)!}}{(x+1)\,x!\,\cancel{(n-x-1)!}} \quad \frac{p}{q}$$

$$= \frac{n-x}{x+1} \quad \frac{p}{q}$$

$$\therefore \frac{P(x+1)}{P(x)} = \frac{n-x}{x+1} \quad \frac{p}{q}$$

$$\Rightarrow P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \, P(x)$$

Proved

Properties of Binomial distribution :-

1. The Binomial distribution is a discrete distribution where the random variable takes the values $0, 1, 2, \dots, n$.

2. The Binomial distribution has two parameters 'n' and 'p' or 'q'.

3. The mean of the binomial distribution is np and variance is npq, standard deviation $\sqrt{npq}$

4. The mean is greater than the variance.

Problems :-

1. Five fair coins are tossed. Find the probability of i) Exactly 3 heads
ii) At least 3 heads.

Sol: Let X be a random variable which denotes the number of heads.

Given, $n = 5$

probability of getting a head, $p = \frac{1}{2}$.

" " " " tail, $q = \frac{1}{2}$

(i).

Prob (exactly 3 heads)

$$\therefore P(X=3) = {}^5C_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^{5-3}$$

$$= 10 \times \frac{1}{8} \times \frac{1}{4}$$

$$= \frac{5}{16}$$

(ii). P(At least 3 heads)

$$\therefore P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$$

$$= \frac{5}{16} + {}^5C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^{5-4} + {}^5C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^{5-5}$$

$$= \frac{5}{16} + \frac{5}{32} + \frac{1}{32}$$

$$= \frac{10+5+1}{32}$$

$$= \frac{16}{32}$$

$$= \frac{1}{2}$$ //

Scanned with CamScanner

2. The mean of a binomial distribution is 6 and the standard deviation is given by $\sqrt{\frac{3}{2}}$. Find the distribution.

Solⁿ: Given,

Mean, $np = 6 \longrightarrow (1)$

Standard deviation $= \sqrt{\frac{3}{2}}$

Variance, $npq = \frac{3}{2} \longrightarrow (2)$

Now, dividing (2) by (1)

$$\frac{npq}{np} = \frac{3/2}{6}$$

$\Rightarrow q = \frac{1}{4}$

$\therefore p = 1 - q$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

We have

$np = 6$

$\Rightarrow n = \frac{6}{p}$

$= \frac{6}{\frac{3}{4}}$

$= 8$

$\therefore$ The Required distribution is

$$p(\tilde{X} = x) = {}^{8}C_{x} \left(\frac{3}{4}\right)^{x} \left(\frac{1}{4}\right)^{8-x}, \quad x = 0, 1, 2, \ldots, n$$

③. If $X \sim B(n,p)$. Show that

$$E\left(\frac{X}{n} - p\right)^2 = \frac{pq}{n}.$$

<u>Sol:</u> Since $X \sim B(n,p)$

$E(X) = np$

$V(X) = npq$

$E\left(\frac{X}{n}\right) = \frac{1}{n} E(X) = \frac{1}{n} \times np = p$

$E(X^2) = V(X) + \{E(X)\}^2$

$= npq + n^2p^2$

We have, $E\left(\frac{X}{n} - p\right)^2$

$= E\left[\frac{X^2}{n^2} - 2.p.\frac{X}{n} + p^2\right]$

$= E\left(\frac{X^2}{n^2}\right) - 2p.E\left(\frac{X}{n}\right) + E(p^2)$

$= \frac{1}{n^2} E(X^2) - 2p.p + p^2$

$= \frac{1}{n^2}.(npq + n^2p^2) - 2p^2 + p^2$

$= \frac{pq}{n} + \cancel{p^2} - \cancel{p^2}$

$= \frac{pq}{n}$ //

Scanned with CamScanner