B. Sc 2nd Semestere

(Hon. and Reg.)

Unit : III

Magnetism

* Divergence and curl of magnetic field from Biot - Savart law : *

∂ Divergance of magnetic field :

According to Biot - Savart law —

$$\vec{B} = \frac{\mu_0}{4\pi} \int \vec{J} \times \frac{\vec{r}}{r^3} \, dV$$

Where , $\vec{J}$ = Volume current density .

Now, $div. \vec{B} = \vec{\nabla} . \vec{B}$

$$= \frac{\mu_0}{4\pi} \int \vec{\nabla} . \left(\vec{J} \times \frac{\vec{r}}{r^3} \right) dV$$

We know that ,

$$\vec{A} . (\vec{B} \times \vec{C}) = \vec{C} . (\vec{A} \times \vec{B}) - \vec{B} . (\vec{A} \times \vec{C})$$

Now,

$$\vec{\nabla} . \vec{B} = \frac{\mu_0}{4\pi} \int \left\{ \frac{\vec{r}}{r^3} . (\vec{\nabla} \times \vec{J}) - \vec{J} . \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right) \right\} dV$$

For steady current , J i.e current density is constant .

$$\therefore \vec{\nabla} \times \vec{J} = 0 .$$

$$\vec{\nabla} . \vec{B} = - \frac{\mu_0}{4\pi} \int \vec{J} . \left(\vec{\nabla} \times \frac{\vec{r}}{r^3} \right) dV$$

$$= - \frac{\mu_0}{4\pi} \int \vec{J} . \left\{ \vec{\nabla} \times (r^{-3} \vec{r}) \right\} dV$$

$$\because \vec{\nabla} \times (\phi \vec{A}) = (\vec{\nabla} \phi) \times \vec{A} + \phi (\vec{\nabla} \times \vec{A})$$

$$\therefore \vec{\nabla} . \vec{B} = - \frac{\mu_0}{4\pi} \int \vec{J} . \left[\vec{\nabla}(r^{-3}) \times \vec{r} + r^{-3} (\vec{\nabla} \times \vec{r}) \right] dV$$

$$\vec{\nabla} . \vec{B} = - \frac{\mu_0}{4\pi} \int \vec{J} . (-3 r^{-3-2} \vec{r} \times \vec{r} + r^{-3} \times 0) \, dV$$

$\left[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \right.$
$\left. \vec{\nabla} \times \vec{r} = 0 \right.$

$$\vec{\nabla}(r^n) = nr^{n-2}\vec{r}$$

$\because \vec{r} \times \vec{r} = 0.$

$\therefore \vec{\nabla} \cdot \vec{B} = 0$

i.e the diverence of magnetic field is zero.

§ Curl of magnetic field :

From the Biot - Savart law —

$$\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J} \times \frac{\vec{R}}{R^3}\, dV$$

We can write,

$$\vec{\nabla}\left(\frac{1}{R}\right) = \nabla(R^{-1}) = -1R^{-1-2}\vec{R}$$
$$= -1R^{-3}\vec{R}$$
$$= -\frac{\vec{R}}{R^3}$$

$$\therefore \left(\frac{\vec{R}}{R^3}\right) = -\vec{\nabla}\left(\frac{1}{R}\right)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int -\vec{J} \times \vec{\nabla}(1/R)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla} \times \left(\frac{\vec{J}}{R}\right) dV$$

$$= \vec{\nabla} \times \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R}\, dV$$

[$\because \vec{\nabla} \times \frac{\vec{J}}{R} = \vec{\nabla}\left(\frac{1}{R}\right) \times \vec{J} + (\vec{\nabla} \times \vec{J})\frac{1}{R}$

For steady current,

$J =$ constant

$\therefore \vec{\nabla} \times \vec{J} = 0.$]

Now, $\text{curl}\,\vec{B} = \vec{\nabla} \times \vec{B}$

$$= \vec{\nabla} \times \vec{\nabla} \times \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R}\, dV$$

Now, we know that

curl of curl of $A = \nabla \times \nabla \times A$

$$= \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 A$$

$$\vec{\nabla} \times \vec{B} = \underbrace{\vec{\nabla}\left(\vec{\nabla}\cdot\left(\frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R}\, dV\right)\right)}_{I_1} - \underbrace{\nabla^2\left(\frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R}\, dV\right)}_{I_2}.$$

$$\therefore \vec{\nabla} \times \vec{B} = I_1 - I_2.$$

$$I_1 = \vec{\nabla}\left(\vec{\nabla}\cdot\left(\frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R}\, dV\right)\right)$$

$$= \vec{\nabla}\left(\frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\frac{\vec{J}}{R}\, dV\right)$$

$$= \vec{\nabla}\left(\frac{\mu_0}{4\pi}\int \vec{J}\,\vec{\nabla}\cdot\left(\frac{1}{R}\right) dV\right.$$

$$= -\vec{\nabla}\left(\frac{\mu_0}{4\pi}\int \vec{J}\,\vec{\nabla}'(1/R)\, dV\right.$$

$$= -\vec{\nabla}\left(\frac{\mu_0}{4\pi}\int \vec{\nabla}'\cdot\left(\frac{\vec{J}}{R}\right) dV\right.$$

$$= -\vec{\nabla}\left(\frac{\mu_0}{4\pi}\oint_S \frac{\vec{J}}{R}\, ds\right) \quad [\text{From divergence theorem}]$$

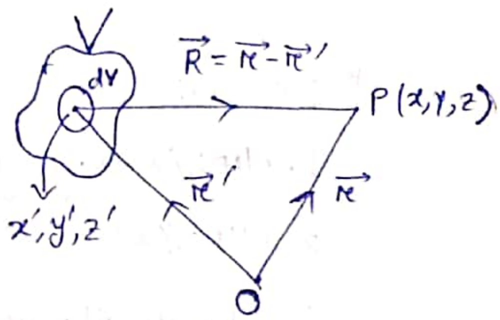

fig: 1

if $R \to \infty$, the current density becomes zero, i.e $\vec{J} = 0$

$$I_1 = 0$$

Now, $I_2 = \nabla^2\left(\frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R}\, dV\right)$

$$= \frac{\mu_0}{4\pi}\int \nabla^2\left(\frac{\vec{J}}{R}\, dV\right)$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}\,\nabla(1/R)\, dV$$

$$= \frac{\mu_0}{4\pi} \int \vec{J} \left(-4\pi \partial(\vec{r}-\vec{r}')\right) dV \qquad \left[\because \nabla^2(1/R) = -4\pi \partial(\vec{r}-\vec{r}')\right]$$

$$= \frac{\mu_0}{4\pi} \cdot \vec{J}(-4\pi) \qquad \left[\because \vec{r}' = \vec{r}\right]$$

$$= -\mu_0 \vec{J}.$$

$$\therefore I_2 = -\mu_0 \vec{J}.$$

i.e curl of magnetic field is —

$$\vec{\nabla} \times \vec{B} = 0 - (-\mu_0 \vec{J})$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

References:

1. http://www.google.com.in
2. http://www.wikipedia.com
3. "Electricity and magnetism" — D. CHATTOPADHYAY, P.C. RAKSHIT