Poisson Distribution

Poisson distribution was discovered by the French mathematician and physicist Simeon Denis Poisson (1781–1840) who published it in 1837. Poisson distribution is a limiting case of binomial distribution under the following conditions.

(i). n, the number of trials is indefinitely large ie $n \to \infty$.

(ii). p, the constant probability of success for each trial is indefinitely small ie $p \to 0$

(iii). $np = \lambda$ (say) is finite.

Thus $p = \lambda/n$, $q = 1 - \lambda/n$, where λ is a positive real numbers.

We know

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)(n-x)!}{x!\,(n-x)!} \cdot \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!\left(1-\frac{\lambda}{n}\right)^x} \cdot \frac{\lambda^x}{n^x} \cdot \left(1-\frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty} P(x) = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!\left(1-\frac{\lambda}{n}\right)^x} \lim_{n\to\infty} \lambda^x \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{1}{x!\,1}\,\lambda^x e^{-\lambda} \qquad \left[\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda},\ \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x = 1\right]$$

$$= \frac{e^{-\lambda}\cdot\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty.$$

Remark:

(1). Poisson distribution can also be derived using Stirling's approximation.

(2). The corresponding distribution function is

$$F(x) = P(X \leq x)$$

$$= \sum_{r=0}^{x} p(r)$$

$$= \sum_{r=0}^{x} \frac{e^{-\lambda} \lambda^{r}}{r!}$$

$$= e^{-\lambda} \sum_{r=0}^{x} \frac{\lambda^{r}}{r!}$$

(3). Following are the some instances where Poisson distribution may be successfully employed:

(i). Number of suicides reported in a particular city.

(ii) Number of faulty blades in a pack of 100.

(iii). Number of printing mistakes at each page of the book.

(iv). Number of air accidents in some unit of time.

Moments of the Poisson distribution :-

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \cancel{x} \cdot \frac{e^{-\lambda}\lambda^x}{\cancel{x}(x-1)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \lambda \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} \quad \left[\sum \frac{\lambda^{x-1}}{(x-1)!} = \left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right) = e^{\lambda} \right.$$

$$= \lambda$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{\infty} \{x(x-1)+x\} p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)\, p(x) + \sum_{x=0}^{\infty} x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \lambda$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x-2}}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda} \cdot \lambda^2 e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$\mu_3' = E(x^3)$$

$$= \sum_{x=0}^{\infty} x^3\, p(x)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\}\, p(x) + \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)\, p(x) + \sum_{x=0}^{\infty} 3x(x-1)\, p(x) + \lambda$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda}\lambda^{x}}{x!} + 3[\lambda^2] + \lambda$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(x^4)\, p(x)$$

$$= \sum_{x=0}^{\infty} x^4\, p(x)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are now obtained as follows

$$\mu_2 = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= (\lambda^3 + 3\lambda^2 + \lambda) - 3(\lambda^2 + \lambda).\lambda + 2\lambda^3$$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda^3 - 3\lambda.^2 + 2\lambda^3$$

$$= \lambda.$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 3\lambda^2 + \lambda.$$

Coefficient of Skewness and Kurtosis :-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = \underline{3 + \frac{1}{\lambda}}$$ (Measure of Kurtosis)

$$\gamma_2 = \beta_2 - 3 \Rightarrow 3 + \frac{1}{\lambda} - 3 = \frac{1}{\lambda}$$

Hence the Poisson distribution is a Skewed distribution.

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Recurrence Relation for moments of the Poisson Distribution :-

By defination,

$$\mu_r = E[x - E(x)]^r$$

$$= \sum_{x=0}^{\infty} (x - E(x))^r P(x)$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

Differentiating w.r.t λ

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^x}{x!} \cdot r(x-\lambda)^{r-1}(-1) + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!} \left\{ -e^{-\lambda} \cdot \lambda^x + x\lambda^{x-1} e^{-\lambda} \right\}$$

$$= \sum_{x=0}^{\infty} -r(x-\lambda)^{r-1} \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!} \left\{ \lambda^{x-1} e^{-\lambda} (x - \lambda) \right\}$$

$$= -r \sum_{x=0}^{\infty} (x-\lambda)^{r-1} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^{r+1}}{x!} \lambda^{x-1} e^{-\lambda}$$

$$= -r \sum_{x=0}^{\infty} (x-\lambda)^{r-1} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^{r+1}}{x!} \cdot \lambda^x \cdot \frac{1}{\lambda} e^{-\lambda}$$

$$= -r \sum_{x=0}^{\infty} (x-\lambda)^{r-1} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \frac{1}{\lambda} \sum_{x=0}^{\infty} (x-\lambda)^{r+1} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= -r\mu_{r-1} + \frac{1}{\lambda} \mu_{r+1}$$

$$\Rightarrow \frac{d\mu_r}{d\lambda} = -r\mu_{r-1} + \frac{1}{\lambda} \mu_{r+1}$$

$$\therefore \mu_{r+1} = \lambda \left\{ r\mu_{r-1} + \frac{d\mu_r}{d\lambda} \right\}.$$

$$\mu_{r+1} = r\lambda\mu_{r-1} + \lambda \cdot \frac{d\mu_r}{d\lambda}.$$

Putting $r = 1, 2, 3$ successively, we get

$$\mu_2 = 1 \cdot \lambda\mu_0 + \lambda\frac{d\mu_1}{d\lambda} = \lambda.$$

$$\mu_3 = 2\lambda\mu_1 + \lambda\frac{d\mu_2}{d\lambda} = \lambda.$$

$$\mu_4 = 3\lambda\mu_2 + \lambda\frac{d\mu_3}{d\lambda} = 3\lambda^2 + \lambda.$$

Moments of Poisson Distribution:-

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \cdots\right\}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{-\lambda + \lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

Remark:

If m is the mean and σ is the s.d of poisson distribution with parameter λ. then the value of $m\,\sigma\,\gamma_1\,\gamma_2$.

Sol: $m.\sigma.\gamma_1.\gamma_2 = m.\sqrt{m}.\frac{1}{\sqrt{m}}.\frac{1}{m}$.

$= 1$.

Additive or reproductive property of independent poisson variates :-

Sum of independent poisson variates is also Poisson variate. More elaborately, if X_i $(i = 1,2,\ldots,n)$ are independent Poisson variates with parameters λ_i; $i = 1,2\ldots,n$, respectively then $\sum_{i=1}^{n} X_i$ is also a Poisson Variate with parameter $\sum_{i=1}^{n} \lambda_i$.

Proof:- $M_{X_i}(t) = e^{\lambda_i(e^t-1)}$, $i = 1,2\cdots n$, $[\because X_i \sim P(\lambda)]$

$$M_{X_1+X_2+\cdots X_n}(t) = M_{X_1}(t).M_{X_2}(t)\cdots M_{X_n}(t)$$

$$= e^{\lambda_1(e^t-1)}.e^{\lambda_2(e^t-1)}\cdots e^{\lambda_n(e^t-1)}$$

$$= e^{(e^t-1)(\lambda_1+\lambda_2+\cdots+\lambda_n)}$$

$$= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)}$$

Which is the mgf of poisson distribution with parameter $(\lambda_1+\lambda_2+\cdots+\lambda_n)$. Hence by Uniqueness theorem on m.g.f's $\sum_{i=1}^{n} X_i$ is also a Poisson variate with Parameter $\sum_{i=1} \lambda_i$

Remark:-

(1) The converse is also true. i.e. If $x_1, x_2, x_3, \dots, x_n$ are independent and $\sum_{i=1}^{n} x_i$ has a poisson distribution, then each of the random variables $x_1, x_2, \dots, x_n$ has a poisson distribution.

Proof:- Let x_1 and x_2 be independent, then

$$x_1 \sim P(\lambda_1) \text{ and } x_2 \sim P(\lambda_2).$$

$\because$ Since x_1 and x_2 are independent.

$$M_{x_1+x_2}(t) = M_{x_1}(t) \cdot M_{x_2}(t)$$

$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

(2). The difference of two independent Poisson variate is not a Poisson variate.

$$M_{x_1-x_2}(t) = M_{x_1+(-x_2)}(t)$$

$$= M_{x_1}(t) \cdot M_{(-x_2)}(t)$$

$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^{-t}-1)}$$

which cannot be put in the form $e^{\lambda(e^t-1)}$. Hence (x_1-x_2) is not a Poisson variate.

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Probability Generating Function of Poisson Distribution :-

$$P.G.F = E[s^x]$$

$$= \sum_{x=0}^{\infty} s^x p(x)$$

$$= \sum_{x=0}^{\infty} s^x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(s\lambda)^x}{x!}$$

$$= e^{-\lambda} \left[1 + s\lambda + \frac{(s\lambda)^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \cdot e^{\lambda s}$$

$$= e^{-\lambda + \lambda s}$$

$$= e^{\lambda(s-1)} \, . \, //$$

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Q: A poisson distribution has a double mode at $x=1$ and $x=2$. What is the probability that x will have one or the other of these two values.

Sol^n:- We know that if the Poisson distribution is bimodal, then the two modes are at the points $x=\lambda$ and $x=\lambda-1$, where λ is the parameter of the Poisson distribution. Therefore, since we are given that the two modes are at the points $x=1$ and $x=2$, then $\lambda=2$.

$$\therefore P(X=1) = \frac{e^{-\lambda} \cdot \lambda}{1!} = e^{-2} \cdot 2.$$

$$P(X=2) = \frac{e^{-\lambda} \cdot \lambda^2}{2!} = \frac{e^{-2} \cdot 2^2}{2} = 2e^{-2}$$

$\therefore$ Required probability

$$P(X=1) + P(X=2)$$

$$= 2e^{-2} + 2e^{-2}$$

$$= 4e^{-2}$$

$$= 0.542$$

Q: If X is a Poisson distribution such that

$$P(X=2) = 9P(X=4) + 90\,P(X=6). \longrightarrow (*)$$

find (i) λ (ii) the mean of X (iii) β_1 the coefficient.

Sol:- If X is a Poisson variate with parameter λ, then

$$P(X=x) = \frac{e^{-\lambda}\cdot\lambda^x}{x!}, \quad x=0,1,\cdots\infty.$$

Hence from (*)

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

$$\Rightarrow \frac{e^{-\lambda}\cdot\lambda^2}{2!} = \frac{9\cdot e^{-\lambda}\cdot\lambda^4}{4!} + \frac{90\cdot e^{-\lambda}\cdot\lambda^6}{6!}$$

$$\Rightarrow \frac{e^{-\lambda}\cdot\lambda^2}{2!} = e^{-\lambda}\left[\frac{9\lambda^4}{4!} + \frac{90\lambda^6}{6!}\right]$$

$$\Rightarrow \frac{\cancel{e^{-\lambda}}\cdot\lambda^2}{2} = \frac{9\lambda^4}{24} + \frac{9\cancel{0}\lambda^6}{72\cancel{0}}$$

$$\Rightarrow 1 = \frac{9\lambda^2}{12} + \frac{9\cancel{0}\lambda^4}{36}$$

$$\Rightarrow 1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\Rightarrow 4 = 3\lambda^2 + \lambda^4$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

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$\Rightarrow \lambda^4 + (4-1)\lambda^2 - 4 = 0$

$\Rightarrow \lambda^4 + 4\lambda^2 - \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2(\lambda^2+4) - 1(\lambda^2+4) = 0$

$\Rightarrow (\lambda^2+4)(\lambda^2-1) = 0$

Either $\lambda^2+4=0, \Rightarrow \lambda^2 = -4$ (NA)

$\lambda^2 - 1 = 0 \Rightarrow \lambda^2 = \pm 1$

$\therefore \lambda = 1.$

$\therefore$ i) Mean $= \lambda = 1$

ii) $\beta_1 =$ Coefficient of Sqewness is $\frac{1}{\lambda} = \frac{1}{1} = 1$

Q: If X is a poisson variate with mean m, show that $\frac{X-m}{\sqrt{m}}$ is a variable with mean zero and variance unity. Find the mgf for this variable and show that it approaches $\exp(t^2/2)$ as $m \to \infty$. Also interpret the result.

Sol:

Let $Y = \frac{X-m}{\sqrt{m}}$,

$$E(Y) = \frac{1}{\sqrt{m}} E(X-m) = 0$$ [Sum of deviation taken from AM is zero].

$$V(Y) = \frac{1}{m} E(X-m)^2$$

$$= \frac{1}{m} \cdot \mu_2$$

$$= \frac{1}{m} \cdot m = 1.$$

Hence $Y = \frac{X-m}{\sqrt{m}}$ has zero mean and unit variance.

M.G.F of $Y = M_Y(t) = E[e^{tY}]$

$$= E\left[e^{t\left\{\frac{X-m}{\sqrt{m}}\right\}}\right]$$

$$= E\left[e^{\frac{tX-tm}{\sqrt{m}}}\right]$$

$$= e^{-\frac{tm}{\sqrt{m}}} E\left[e^{\frac{tx}{\sqrt{m}}}\right]$$

$$= e^{-\frac{tm}{\sqrt{m}}} \sum_{x=0}^{\infty} e^{\frac{tx}{\sqrt{m}}} \cdot \frac{e^{-m} m^x}{x!}$$

$$= e^{-tm/\sqrt{m}} \cdot e^{-m} \sum_{x=0}^{\infty} \frac{(me^{t/\sqrt{m}})^x}{x!}$$

$$= e^{-t\sqrt{m}-m} \sum_{x=0}^{\infty} \frac{(me^{t/\sqrt{m}})^x}{x!}$$

$$= e^{-m-t\sqrt{m}} \cdot \left\{ 1 + me^{t/\sqrt{m}} + \frac{(me^{t/\sqrt{m}})^2}{2!} + \cdots \right\}$$

$$= e^{-m-t\sqrt{m}} \cdot e^{me^{t/\sqrt{m}}}$$

$$= e^{-m-t/\sqrt{m} + me^{t/\sqrt{m}}}$$

$$= \exp\left[-m - t\sqrt{m} + m\left\{ 1 + \frac{t}{\sqrt{m}} + \frac{t^2}{2!m} + \cdots \right\} \right]$$

$$\lim_{m\to\infty} M_Y(t) = e^{t^2/2} \;//$$

Q: If X and Y are independent Poisson variates, show that the conditional distribution of X given $X+Y$, is binomial.

Soln: Let X and Y be independent Poisson variates with parameters λ and μ respectively. Then $X+Y$ is also a Poisson variate with parameter $\lambda+\mu$.

$$P\left[X=r / (X+Y)=n\right] = \frac{P\left[X=r \cap X+Y=n\right]}{P(X+Y=n)}$$

$$= \frac{P(X=r \cap Y=n-x)}{P(X+Y=n)}$$

$$= \frac{P(X=r \cap Y=n-r)}{P(X+Y=n)}$$

$$= \frac{P(X=r)\cdot P(Y=n-r)}{P(X+Y=n)}$$ [$\because$ X and Y are independent]

$$= \frac{e^{-\lambda}\cdot\frac{\lambda^r}{r!}\cdot e^{-\mu}\frac{\mu^{n-r}}{(n-r)!}}{e^{-(\lambda+\mu)}\cdot\frac{(\lambda+\mu)^n}{n!}}$$

$$= \frac{\cancel{e^{-\lambda}}\lambda^r}{r!}\cdot\frac{\cancel{e^{-\mu}}\mu^{n-r}}{(n-r)!}\cdot\frac{n!}{e^{-(\lambda+\mu)}\cdot(\lambda+\mu)^n}$$

$$= \frac{n!}{r!\,(n-r)!}\cdot\frac{\lambda^r\mu^{n-r}}{(\lambda+\mu)^n}$$

$$= \frac{n!}{r!\,(n-r)!}\cdot\frac{\lambda^r}{(\lambda+\mu)^r}\cdot\frac{\mu^{n-r}}{(\lambda+\mu)^{n-r}}$$

$$= \frac{n!}{r!\,(n-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

$$= {}^nC_r\cdot\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{n-r}, \text{ where } p=\frac{\lambda}{\lambda+\mu},$$

Hence the conditional distribution of X given $X+Y=n$, is a binomial distribution with parameters n and $p=\frac{\lambda}{\lambda+\mu}$.

Q: If X is a poisson variate with parameter m and μ_r is the r^{th} central moment,

Prove that

$$m\left({}^{r}C_1\,\mu_{r-1} + {}^{r}C_2\,\mu_{r-2} + \dots + {}^{r}C_r\,\mu_0\right) = \mu_{r+1}$$

Sol: Since $X \sim P(m)$, its probability function is given by

$$P(x) = \frac{e^{-m} \cdot m^x}{x!}, \quad x = 0, 1, 2, \dots, m.$$

By defination:

$$\mu_{r+1} = E\left[X - E(X)\right]^{r+1}$$

$$= E\left[X - m\right]^{r+1}$$

$$= \sum_{x=0}^{\infty} (x-m)^{r+1} \cdot P(x)$$

$$= \sum_{x=0}^{\infty} (x-m)^{r} (x-m) \cdot \frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x (x-m)^r e^{-m} \cdot m^x}{x!} - m \sum_{x=0}^{\infty} (x-m)^r \frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x (x-m)^r e^{-m} \cdot m^x}{x!} - m\mu_r$$

let $x - 1 = y$ $\Rightarrow x = y + 1$

$$= \sum_{x=0}^{\infty} \frac{x (x-m)^r e^{-m} \cdot m^x}{x (x-1)!} - m\mu_r$$

$$= \sum_{x=1}^{\infty} \frac{(x-m)^r e^{-m} \cdot m^x}{(x-1)!} - m\mu_r$$

Let $x-1=y$

$\Rightarrow x=y+1.$

$$\mu_{r+1}=\sum_{y=0}^{\infty}\frac{(y-m+1)^r e^{-m} m^{y+1}}{y!}-m\mu_r$$

$$=\sum_{y=0}^{\infty}\frac{(y-m+1)^r e^{-m}\cdot m^y\cdot m}{y!}-m\mu_r$$

$$=m\sum_{y=0}^{\infty}(y-m+1)^r\frac{e^{-m}\cdot m^y}{y!}-m\mu_r$$

$$=m\sum_{y=0}^{\infty}(y-m+1)^r\cdot P(y)-m\mu_r$$

$$=m\sum_{y=0}^{\infty}\left[(y-m)^r+{}^rC_1(y-m)^{r-1}+\cdots+{}^rC_{r-1}(y-m)+1\right]P(y)$$

$$-m\mu_r.$$

$$=m\left(\mu_r+{}^rC_1\mu_{r-1}+\cdots+{}^rC_r\mu_0\right)-m\mu_r$$

$$=m\mu_r+m\left({}^rC_1\mu_{r-1}+{}^rC_2\mu_{r-2}+\cdots+{}^rC_r\mu_0\right)-\cancel{m\mu_r}$$

$$=\left({}^rC_1\mu_{r-1}+{}^rC_2\mu_{r-2}+\cdots+{}^rC_r\mu_0\right)$$ //

Recurrence formula for the probability of Poisson distribution :-

For a poisson distribution with parameter λ, we have —

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \dots, \infty$$

$$P(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}, \quad x = 0, 1, 2, \dots \infty.$$

$$\therefore \frac{P(x+1)}{P(x)} = \frac{\lambda}{x+1}$$

$$\Rightarrow P(x+1) = \frac{\lambda}{(x+1)} P(x). \Rightarrow P(x) =$$

//

$$\frac{P(k)}{P(k-1)} = \frac{\lambda}{k}$$

$$P(k) = \frac{\lambda}{k} P(k-1)$$

//

Recurrence Relation of moments about mean :-

$$\mu_{r+1} = r\lambda\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda}$$