Negative Binomial distribution :-

Introduction:

The equality of the mean and variance is an important characteristics of the Poisson distribution, whereas for the binomial distribution the mean is always greater than the variance. Occasionally, however, observable phenomenon give arise to empirical discrete distributions which show a variance to larger than the mean. Some of the commonest examples of such behaviour are the frequency distributions of the plants density obtained by quadrant sampling when the clustering of plants makes the simple Poisson model inapplicable. It has been shown by different investigators that in such cases the negative binomial distribution (inverse of binomial distribution) provides an excellent model because this distribution has a variance larger than the mean. For eg: deaths of insects, number of insect bites leads to negative binomial distribution.

The negative binomial distribution can be derived from many empirical considerations in many ways. Here we will consider the binomial probability situation with some modifications.

Suppose we have a succession of n bernoulli trials. we assume that

(i) the trials are independent.

(ii) the probability of success 'p' in a trial remains constant from trial to trial.

Let $f(x; r, p)$ denote the probability that there are x failures preceding the r^{th} success in $(x+r)$ trials. Now the last trial may be success, whose probability is p. In the remaining $(x+r-1)$ trials are must have $(r-1)$ success whose probability is given by the binomial probability law by the expression.

$$\binom{x+r-1}{r-1} p^{r-1} q^{x}.$$

Therefore by compound probability theorem $f(x; r, p)$ is given by the product of these two probabilities.

$$f(x;r,p) = \binom{x+r-1}{r-1} p^{r-1} q^{x} \cdot p = \binom{x+r-1}{r-1} p^{r} \cdot q^{x}.$$

Definition:

A random variable X is said to follow a negative binomial distribution with parameters r and p if its probability mass function is given by.

$$P(X=x) = \binom{x+r-1}{r-1} p^{r} q^{x}, \quad x = 0, 1, 2, \ldots \longrightarrow (1)$$

Also $\binom{x+r-1}{r-1} = \binom{x+r-1}{x}$ $\left[\because \binom{n}{r} = \binom{n}{n-r}\right]$

$$= \frac{(x+r-1)(x+r-2)\cdots(r+1)\, r\, (r-1)!}{x!\,(r-1)!}$$

$$= \frac{(-1)^x (-r)(-r-1)\cdots(-r-x+2)(-r-x+1)}{x!}$$

$$= (-1)^x \binom{-r}{x}$$

from ①.

$$P(x) = \binom{-r}{x} p^r (-q)^x \quad x = 0, 1, 2 \ldots,$$

If $p = \frac{1}{Q}$ and $q = \frac{P}{Q}$, so that, $Q - P = 1$,
$[\because p+q=1]$

$$P(x) = \left\{ \binom{-r}{x} \left(\frac{1}{Q}\right)^r \left(-\frac{P}{Q}\right)^x \right.$$

$$= \left\{ \binom{-r}{x} Q^{-r} (-P/Q)^x \right.$$

This is the general term in the negative binomial expansion $(Q-P)^{-r}$

Some important deductions:-

a) Geometric distribution: If we put $r=1$ in equation we'll get

$$P(x) = \left\{ \binom{x+1-1}{1-1} \cdot p^1 q^x \right.$$

$$= \binom{x}{0} p q^x$$

$$= p q^x$$

Hence, negative binomial distribution may be regarded as geometric distribution.

Moment Generating function of Negative Binomial Distribution.

$$M_x(t) = E[e^{tx} p(x)]$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \binom{-r}{x} Q^{-r} \left(\frac{-P}{Q}\right)^{x}$$

$$= \sum_{x=0}^{\infty} \binom{-r}{x} Q^{-r} \left(\frac{-P}{Q} e^{t}\right)^{x}$$

$= (Q - Pe^{t})^{-r}$ which is the mgf of negative binomial distribution.

$$\mu_1' = \frac{d}{dt} M_x(t) \Big/_{t=0}$$

$$\left\{ P = \frac{1}{Q}, \; q = \frac{P}{Q} \right.$$

$$P = 2Q$$

$$= \frac{2P}{P}$$

$$= \frac{d}{dt} (Q - Pe^{t})^{-r} \Big/_{t=0}$$

$$= -r (Q - Pe^{t})^{-r-1} \{-Pe^{t}\} \Big/_{t=0}$$

$$= +r Pe^{t} (Q - Pe^{t})^{-r-1} \Big/_{t=0}$$

$$= r Pe^{0} (Q - Pe^{0})^{-r-1}$$

$$= r P (Q - P)^{-r-1}$$

$$= r P \quad [\because Q - P = 1]$$

$$= \frac{rq}{P}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left\{\frac{d}{dt} M(t)\right\}\Big|_{t=0}$$

$$= \frac{d}{dt}\left[rPe^t (Q - Pe^t)^{-r-1}\right]\Big|_{t=0}$$

$$= rPe^t \cdot (-r-1)(Q-Pe^t)^{-r-1-1} \cdot (-Pe^t) + (Q-Pe^t)^{-r-1} \cdot rpe^t$$

$$= rpe^t (-r-1)(Q-Pe^t)^{-r-2}(-Pe^t) + rPe^t (Q-Pe^t)^{-r-1}\Big|_{t=0}$$

$$= rP(-r-1)(Q-P)^{-r-2}(-P) + rP(Q-P)^{-r-1}$$

$$= rP(-r-1)(-P) + rP$$

$$= rP[-1(r+1)](-P) + rP$$

$$= rP^2(r+1) + rP$$

Now

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= rP^2(r+1) + rP - r^2P^2$$

$$= \cancel{r^2P^2} + rP^2 + rP - \cancel{r^2P^2}$$

$$= rP(P+1)$$

$$= rPQ,$$

$$= r\frac{q}{p}\frac{1}{p} = \frac{rq}{p^2}$$

$P = \frac{1}{Q}$

$Q = \frac{1}{P}$

$q = \frac{P}{Q}$

$qQ = P$

$\frac{q}{p}$

$P = \frac{1}{Q} \Rightarrow Q = \frac{1}{P}$

$q = \frac{P}{Q}$

$\Rightarrow P = qQ =$

$PQ = qQ.$

$= q/p$

Poisson distribution as a case of limiting case of Negative Binomial distribution:-

Negative binomial distribution tends to Poisson distribution as $P \to 0$, $r \to \infty$ such that $rP = \lambda$ (finite). Proceeding to the limits we get,

$$\lim p(x) = \lim_{r\to\infty} \binom{x+r-1}{r-1} \cdot p^{r} q^{x}$$

$$= \lim_{r\to\infty} \binom{x+r-1}{x} Q^{-r} \left(\frac{P}{Q}\right)^{x}$$

$$= \lim_{r\to\infty} \frac{(x+r-1)(x+r-2)\cdots(r+1)\,r\,(r-1)!}{x!\,(r-1)!} \cdot Q^{-r}\left(\frac{P}{Q}\right)^{x}$$

$$= \lim_{r\to\infty} \frac{(x+r-1)(x+r-2)\cdots(r+1)r}{x!} Q^{-r}\left(\frac{P}{Q}\right)^{x}$$

$$= \lim_{r\to\infty} \frac{(x+r-1)(x+r-2)\cdots(r+1)r \cdot (1+P)^{-r}\left(\frac{P}{1+P}\right)^{x}}{x!}$$

$$= \lim_{r\to\infty} \frac{r\left(1+\frac{x-1}{r}\right)\left(1+\frac{x-2}{r}\right)\cdots r\left(1+\frac{1}{r}\right)\cdot r \cdot (1+P)^{-r}\left(\frac{P}{1+P}\right)^{x}}{x!}$$

$$= \lim_{r\to 0} \left\{\frac{1}{x!}\left(1+\frac{x-1}{r}\right)\left(1+\frac{x-2}{r}\right)\cdots\left(1+\frac{1}{r}\right)\cdot 1 \cdot r^{x}(1+P)^{-r}\left(\frac{P}{1+P}\right)^{x}\right\}$$

$$= \frac{1}{x!}\lim_{r\to 0}\left\{(1+P)^{-r}\left(\frac{rP}{1+P}\right)^{x}\right\}$$

$$= \frac{1}{x!}\lim_{r\to 0}\left\{\left(1+\frac{\lambda}{r}\right)^{-r}\left(\frac{\lambda}{1+\lambda/r}\right)^{x}\right\}$$

$$= \frac{\lambda^{x}}{x!}\lim_{r\to 0}\left(1+\frac{\lambda}{r}\right)^{-r} \cdot \lim_{r\to 0}\left(1+\frac{\lambda}{r}\right)^{-x}$$

$$= \frac{\lambda^{x}}{x!} e^{-\lambda}\cdot 1 = \frac{e^{-\lambda}\lambda^{x}}{x!}$$ is the pmf of P.D (λ).

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Probability Generating Function :-

Let X be a random variable following negative binomial distribution.

$$P_X(s) = E[s^x]$$

$$= \sum_{x=0}^{\infty} s^x P(x)$$

$$= \sum_{x=0}^{\infty} s^x \binom{-r}{x} p^r (-q)^x$$

$$= \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-qs)^x$$

$$= p^r (1-qs)^{-r}$$

$$= \frac{p^r}{(1-qs)^r}$$

$$= \left[\frac{p}{(1-qs)}\right]^r$$ //.

Recurrence ~~Relation~~ formula :-

Proof:

We have,

$$f(x;r,p) = \binom{x+r-1}{r-1} p^r q^x$$

$$f(x+1,r,p) = \binom{x+r}{r-1} p^r q^{x+1}$$

$$\therefore \frac{f(x+1;r,p)}{f(x;r,p)} = \frac{\binom{x+r}{r-1} p^r q^{x+1}}{\binom{x+r-1}{r-1} p^r q^x}$$

$$= \frac{\dfrac{(x+r)!}{(r-1)!\,(x+\cancel{r}-\cancel{r}+1)!} p^r q^{x+1}}{\dfrac{(x+r-1)!}{(r-1)!\,(x+r-1-r+1)!} p^r q^x}$$

$$= \frac{(x+r)!}{(\cancel{r-1})!\,(x+1)!} \cancel{p^r} q^{x+1} \times \frac{(\cancel{r-1})!(x)!}{(x+r-1)!\,\cancel{p^r} q^x}$$

$$= \frac{(x+r)\,(x+r-1)!\;q}{(x+1)\cdot \cancel{x!}} \times \frac{\cancel{x!}}{(x+r-1)!}$$

$$= \frac{x+r}{x+1} \cdot q$$

$$\therefore f(x+1) = \frac{x+r}{x+1} q \cdot f(x)$$ is the Recurrence formula.