Geometric Distribution:

Defination:

A random variable X is said to have a geometric distribution if it assumes only non-negative values and its probability mass function is given by:

$$P(X=x) = \begin{cases} q^x p, & x = 0, 1, 2 \ldots \\ 0, & \text{Otherwise.} \end{cases}$$

Remark:

(i) The various probabilities for $x = 0, 1, 2, \ldots$ are the terms of geometric progression series, hence the name geometric distribution.

Lack of memory :-

The geometric distribution is said to lack memory in a certain sense. Suppose an event E can occur at one of the times $t = 0, 1, 2, \ldots$ and the occurrence (waiting) time X has a geometric distribution with parameter p.

Thus

$$P(X=t) = q^t p, \quad t = 0, 1, 2 \ldots$$

Suppose we know that the event E has not occurred before k. i.e $X \geq k$. Let $Y = X-k$. Thus Y is the amount of additional time needed for E to occur. We can show that

$$P(Y=t / X \geq k) = P(X=t) = pq^t$$

which implies that the additional time to wait has the same distribution as initial time to wait.

Q: Establish the memoryless property of geometric distribution.

show on

$$P(Y=t / X \geq k) = P(X=t) = pq^t$$

Proof:

we have

$$P(X \geq r) = \sum_{s=r}^{\infty} pq^s$$

$$= p\left(q^r + q^{r+1} + q^{r+2} + \dots\right)$$

$$= pq^r\left(1 + q + q^2 + q^3 + \dots\right)$$

$$= pq^r (1-q)^{-1}$$

$$= \frac{pq^r}{1-q}$$

$$= \frac{pq^r}{p}$$

$$= q^r$$

$$P(Y \geq t / X \geq k) = \frac{P(Y \geq t \cap X \geq k)}{P(X \geq k)}$$

$$= \frac{P(X-k \geq t \cap X \geq k)}{P(X \geq k)}$$

$$= \frac{P(X \geq k+t)}{P(X \geq k)} = \frac{q^{t+k}}{q^k} = q^t$$

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$\therefore\ P(Y=t / X\geq u) = P(Y\geq t / X\geq u) - P(Y\geq t+1 / X\geq u)$

$$= q^t - q^{t+1}$$

$$= q^t(1-q)$$

$$= pq^t$$

$$= P(X=t).$$

Moment of Generating Function :-

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot q^x p$$

$$= p \sum_{x=0}^{\infty} (e^t q)^x$$

$$= p[1 + qe^t + (qe^t)^2 + \cdots]$$

$$= p(1-qe^t)^{-1}$$

$$= \frac{p}{(1-qe^t)}.$$

Raw moments of Geometric distribution.

$$\mu_1' = \frac{d}{dt}\left[M_x\, t\right]_{t=0}$$

$$= \frac{d}{dt}\left(\frac{p}{1-qe^t}\right)$$

$$= \frac{d}{dt}\, p\,(1-qe^t)^{-1}$$

$$= p(-1).\,(1-qe^t)^{-2}.\,(-1)qe^t$$

$$= pqe^t(1-qe^t)^{-2}\Big/_{t=0}$$

$$= pq(1-q)^{-2}$$

$$= \frac{pq}{(1-q)^2}$$

$$= \frac{pq}{p^2}$$

$$= \frac{q}{p}$$

$$\mu_2' = \frac{d}{dt^2}\left[M_x(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left\{pqe^t(1-qe^t)^{-2}\right\}$$

$$= (1-qe^t)^{-2}.\,pqe^t + pqe^t.\,(-2)(1-qe^t)^{-3}.\,(-qe^t)$$

$$= (1-qe^t)^{-2}.\,pqe^t + 2pq^2e^{2t}(1-qe^t)^{-3}$$

$$= (1-q)^{-2}.\,pq + 2pq^2(1-q)^{-3}$$

$$= \frac{1}{p^2}.\,pq + 2pq^2\frac{1}{p^3}$$

$$= \frac{q}{p} + \frac{2q^2}{p^2}$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2}$$

$$= \frac{q}{p} + \frac{q^2}{p^2}$$

$$= \frac{pq + q^2}{p^2}$$

$$= \frac{q(p+q)}{p^2}$$

$$= \frac{q}{p^2}.$$

Hence, mean and variance of geometric distribution is $\frac{q}{p}$, $\frac{q}{p^2}$.

Remark:

(1). Variance is greater than mean:

(2). Pgf is $\frac{p}{1-qs}$

** Q: Let X_1, X_2 be independent r.v's each having geometric distribution $q^k p$, $k=0,1,2\ldots$ Show that the conditional distribution of X_1/X_1+X_2 is uniform.

Sol:

$$P(X_1=r / X_1+X_2=n) = \frac{P(X_1=r \cap X_1+X_2=n)}{P(X_1+X_2=n)}$$

$$= \frac{P(X_1=r \cap X_2=n-x_1)}{P(X_1+X_2=n)}$$

$$= \frac{P(X_1=r \cap X_2=n-r)}{P(X_1+X_2=n)}$$

$$= \frac{P(X_1=r) \cdot P(X_2=n-r)}{\sum_{s=0}^{n} P(X_1=s \cap X_2=n-s)}$$

$$= \frac{P(X_1=r) \cdot P(X_2=n-r)}{\sum_{s=0}^{n} P(X_1=s) \cdot P(X_2=n-s)}$$ [$\because$ X_1 and X_2 are independent]

$$= \frac{pq^r \cdot pq^{n-r}}{\sum_{s=0}^{n} pq^s \cdot pq^{n-s}}$$

$$= \frac{p^2 q^n}{\sum_{s=0}^{n} p^2 q^n} = \frac{p^2 q^n}{p^2 q^n (n+1)}$$

$$= \frac{1}{n+1}, \quad r = 0,1,2\ldots$$

Hence the result.

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Recurrence Relation for moments

$$\mu_r = E[X-\mu_1']^r$$

$$= \sum_{k=0}^{\infty} \left(k - \frac{q}{p}\right)^r q^k p$$

$$\frac{d\mu_r}{dp} = \sum_{k=0}^{\infty} \left(k - \frac{q}{p}\right)^{r-1} r q^{k-1} p (p^{-2}) + \sum_{k=0}^{\infty}\left(k - \frac{q}{p}\right)^r$$

$$\left[k q^{k-1}(-1)p + q^k \right]$$

$$= \frac{r}{p^2} \sum_{k=0}^{\infty} \left(k - \frac{q}{p}\right)^{r-1} q^k p - \sum_{k=0}^{\infty}\left(k - \frac{q}{p}\right)^r q^{k-1}(kp - q)$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\sum_{k=0}^{\infty}\left(k - \frac{q}{p}\right)^{r+1} q^k p$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\mu_{r+1}$$

Then, $\mu_{r+1} = \frac{rq}{p^2}\mu_{r-1} - q\frac{d\mu_r}{dp}, \quad r = 0, 1, 2, \ldots$

Now, $r = 0, 1, 2 \ldots$

$\mu_1 = 0$

$\mu_2 = \frac{q}{p^2}$.