Matrices

A rectangular arrangement of $m \times n$ real numbers in m rows and n columns and enclosed by a pair of brackets [ ] or ( ) is called an $m \times n$ (read as 'm by n') matrix.

The general $m \times n$ matrix is the following:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Here each of $m \times n$ real numbers is called an element or entry of the matrix.

The order of the above matrix A is $m \times n$.

Types of Matrices

① Row Matrix : A matrix containing one row is called a row matrix.

Eg. : $\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$ or $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$ is a column matrix of order 1×3.

② **Column matrix :** A matrix containing only one column is called a column matrix.

Eg. - $\begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}$ or $\begin{pmatrix} 6 \\ -5 \\ 2 \end{pmatrix}$ is a column matrix of order 3×1.

③ **Square matrix :** If number of rows and columns in a matrix are equal then it is called a square matrix.

Eg : $\begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix}$ is a square matrix of order 3×3.

④ **Diagonal matrix :** A square matrix all of whose elements except those in the leading diagonal, are zeroes is called a diagonal matrix.

Eg : $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ a diagonal matrix of order 2.

& $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ a diagonal matrix of order 3.

⑤ Scalar matrix : A diagonal matrix whose all the diagonal elements are equal is called a scalar matrix.

Eg: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 3.

⑥ Unit matrix : A diagonal matrix whose all the diagonal elements are unity is called a unit matrix. The unit matrix of order n is denoted by I_n.

Eg: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ a unit matrix of order 3.

⑦ Null matrix : If all the elements of matrix are zero then it is called a null or zero matrix. It is denoted by O.

Eg: $\underset{2\times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$, $\underset{2\times 2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$, $\underset{3\times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$.

(8) Singular and non-singular matrices :

A matrix is said to be singular if the determinant of the matrix is zero, otherwise it is non-singular matrix.

Eg: (i) $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - 4 \times 1 = 4 - 4 = 0$$

$\therefore$ A is singular.

(ii) $B = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}$

$$|B| = \begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix} = 4 \times (-2) - 0 \times 3 = -8 \neq 0$$

$\therefore$ B is non-singular.

Equality of matrices

Two matrices A and B are said to be equal if they are of same order and their corresponding elements are equal.

Eg: $A = \begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 4^2 & 5^2 & 6^2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{bmatrix}$

are equal matrices.

Thus $A = B$.

Matrix addition and substraction

Let A and B be two matrices of same order their sum is written as A + B, is the matrix whose elements are the sum of the corresponding elements of A and B.

Eg: $A = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+2 & -3+5 \\ 0+1 & 3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Similarly, their substraction is written as A - B.

$$A - B = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -3-5 \\ 0-1 & 3-1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 \\ -1 & 2 \end{bmatrix}$$

Scalar multiplication

The product of the matrix A by a scalar k, written as kA, is the matrix obtained by multiplying each element of A by k.

Eg: $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & 5 \end{bmatrix}$

Then $2A = 2\begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 4 \\ 0 & 6 & 10 \end{bmatrix}$

Matrix multiplication

Let $A = [a_{ij}]_{m \times p}$ and $B = [b_{ij}]_{p \times n}$ are matrices such that the number of columns of A is equal to the number of rows of B. Then the product of A & B is denoted by AB is the $m \times n$ matrix whose ij th entry is obtained by multiplying the ith row of A by the jth column of B. That is

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ \boxed{a_{i1} \; \cdots \; a_{ip}} & & \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \boxed{\begin{matrix} b_{1j} \\ \vdots \\ b_{pj} \end{matrix}} & \cdots & b_{1n} \\ \vdots & & & & \vdots \\ b_{p1} & \cdots & & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & & \vdots \\ & \boxed{c_{ij}} & \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}$$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} \dots + a_{ip} b_{pj}$

$$= \sum_{k=1}^{p} a_{ik} b_{kj}$$

Note:

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & \dots & R_1 C_n \\ R_2 C_1 & R_2 C_2 & \dots & R_2 C_n \\ \vdots & \vdots & \vdots & \vdots \\ R_m C_1 & R_m C_2 & \dots & R_m C_n \end{bmatrix}$$

where $R_1, R_2, \dots, R_m$ are m rows of A and $C_1, C_2, \dots, C_n$ are n columns of B.

Eg: $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & R_1 C_3 \\ R_2 C_1 & R_2 C_2 & R_2 C_3 \end{bmatrix}$$

Now, $R_1 C_1 = (1 \quad 3) \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 1 \times 2 + 3 \times 5 = 17$

$R_1 C_2 = (1 \quad 3) \begin{bmatrix} 0 \\ -2 \end{bmatrix} = 1 \times 0 + 3 \times (-2) = -6$

$R_1 C_3 = (1 \quad 3) \begin{bmatrix} -4 \\ 6 \end{bmatrix} = 1 \times (-4) + 3 \times 6 = -4 + 18$

$= 14$

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$R_2C_1 = (2 \quad -1)\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\times 2 + (-1)\times 5 = 4 - 5 = -1$

$R_2C_2 = (2 \quad -1)\begin{bmatrix} 0 \\ -2 \end{bmatrix} = 2\times 0 + (-1)\times(-2) = 0 + 2 = 2$

$R_2C_3 = (2 \quad -1)\begin{bmatrix} -4 \\ 6 \end{bmatrix} = 2\times(-4) + (-1)\times 6 = -8-6 = -14.$

$\therefore AB = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$

Transpose of a matrix

The matrix obtained from a given matrix by interchanging its rows and columns is called transpose of A and is denoted by A' or A^T.

Eg: $A = \begin{bmatrix} 3 & 4 \\ 5 & -3 \\ 6 & 0 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & 5 & 6 \\ 4 & -3 & 0 \end{bmatrix}$

Adjoint of a square matrix

Let us consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then Determinant of matrix A is

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The matrix formed by the cofactors of the elements in Δ is :

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

The transpose of this matrix i.e.

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

is called the adjoint of the matrix A and is written as adj. A .

Ex ①: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$, verify whether $AB = BA$.

Sol^n

$$AB = \begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{array}{cc} C_1 & C_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}} \end{array}$$

$$= \begin{bmatrix} 1\times5 + 2\times0 & 1\times6 + 2\times(-2) \\ 3\times5 + 4\times0 & 3\times6 + 4\times(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6-4 \\ 15 & 18-8 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix} \begin{array}{cc} C_1 & C_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \end{array}$$

$$= \begin{bmatrix} 5\times1 + 6\times3 & 5\times2 + 6\times4 \\ 0\times1 + (-2)\times3 & 0\times2 + (-2)\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 5+18 & 10+24 \\ -6 & -8 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ -6 & -8 \end{bmatrix}$$

$AB \neq BA$

Verified.

Ex-(2) If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ then find $2A + 3B$.

Solⁿ: $2A = 2\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix}$

$3B = 3\begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$

$2A + 3B = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$

$= \begin{bmatrix} 0+21 & 4+18 & 6+9 \\ 4+3 & 2+12 & 8+15 \end{bmatrix}$

$= \begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$ //

Ex-③ Find x, y, z & t where

$$3\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2t \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+t & 3 \end{bmatrix}$$

Sol.ⁿ

$$\begin{bmatrix} 3x & 3y \\ 3z & 3t \end{bmatrix} = \begin{bmatrix} x+4 & x+y+6 \\ z+t-1 & 2t+3 \end{bmatrix}$$

Equating the corresponding elements, we get

$3x = x+4$,

$\Rightarrow 3x - x = 4$

$\Rightarrow 2x = 4$

$\Rightarrow x = \frac{4}{2} = 2$

$3y = x+y+6$

$\Rightarrow 3y - y = 2+6 = 8$

$\Rightarrow 2y = 8 \Rightarrow y = \frac{8}{2} = 4$

$3z = z+t-1$,

$\Rightarrow 3z - z = t-1$

$\Rightarrow 2z = 3-1$

$\Rightarrow 2z = 2$

$\Rightarrow z = \frac{2}{2} = 1$

$3t = 2t+3$

$\Rightarrow 3t - 2t = 3$

$\Rightarrow t = 3$

Ex-④ Show that the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ satisfies the matrix eqⁿ $A^2 - 5A + 7I = 0$.

Solⁿ: $A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$LHS = A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15 & 5-5 \\ -5-(-5) & 3-10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0 = RHS$$