Inverse of a square matrix

If A and B are two square matrices of the same order n such that $\boxed{AB = BA = I_n}$ where I_n is the unit matrix of order n then each is called the inverse of the other.

B is called the inverse of A and it is denoted by A^{-1}.

Similarly, A is called the inverse of B and it is denoted by B^{-1}.

Theorem: If A is a square matrix of order n, then

$$\boxed{A(\text{Adj. } A) = |A| I_n = (\text{Adj. } A) A}$$

where I_n is the unit matrix of order n.

Expression for the inverse of the matrix A

$$A^{-1} = \frac{\text{Adj. } A}{|A|}, \quad \text{where } |A| \neq 0.$$

Ex: Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

Solⁿ:

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} - 0 + 2\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= 1(-8-3) + 2(2+4)$$

$$= -11 + 12$$

$$= 1$$

$$\neq 0$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 1 & 8 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^T}{1}$$

$$= \frac{\begin{bmatrix} (-8-3) & -(16-12) & +(2+4) \\ -(0-2) & +(8-8) & -(1-0) \\ +(0+2) & -(3-4) & +(-1-0) \end{bmatrix}^T}{1}$$

$$= \begin{bmatrix} -11 & -4 & 6 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{bmatrix}^T$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

Ex: Solve the following system of equations with the help of matrices:

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

Sol^n The given system of equations can be written in matrix form as $AX = B \Rightarrow X = A^{-1}B$

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$= 3(-3+2) - 1(2+1) + 2(4+3)$$

$$= -3 - 3 + 14 = 8 \neq 0.$$

Hence A^{-1} exist.

Now, $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

$$= \frac{1}{8}\begin{bmatrix} +\begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{8}\begin{bmatrix} +(-3+2) & -(2+1) & +(4+3) \\ -(1-4) & +(3-2) & -(6-1) \\ +(-1+6) & -(-3-4) & +(-9-2) \end{bmatrix}^T$$

$$= \frac{1}{8}\begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1\times 3 + 3\times(-3) + 5\times 4 \\ -3\times 3 + 1\times(-3) + 7\times 4 \\ 7\times 3 + (-5)\times(-3) + (-11)\times 4 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 8\times\frac{1}{8} \\ 16\times\frac{1}{8} \\ -8\times\frac{1}{8} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = 1, \quad y = 2, \quad z = -1 \quad //$$

Ex: Find the inverse of (a) $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & 4 \end{bmatrix}$

Ex: Solve by matrix method :

(a) $x + 2y - 4z = -3$

$2x + 6y - 5z = -2$

$3x + 11y - 4z = 12$

(b) $x + y + z = 6$

$x + 2y + 3z = 14$

$-x + y - z = -2$