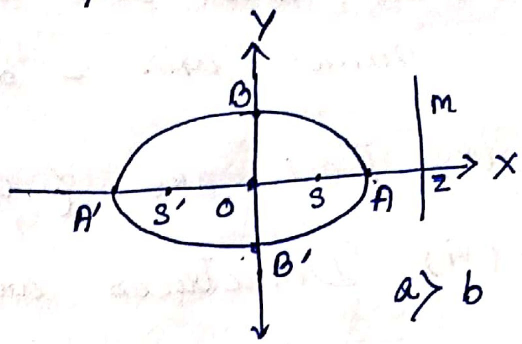
Ellipse

(A) Standard equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \text{ where } a > b$$



(i) Centre is $(0,0)$

(ii) Eccentricity e is

$$b^2 = a^2(1-e^2) \quad , \quad e<1$$

(iii) The major axis is along x- axis & length of the major axis $= 2a$.

(iv) The minor axis is along y-axis & length of the minor axis $= 2b$.

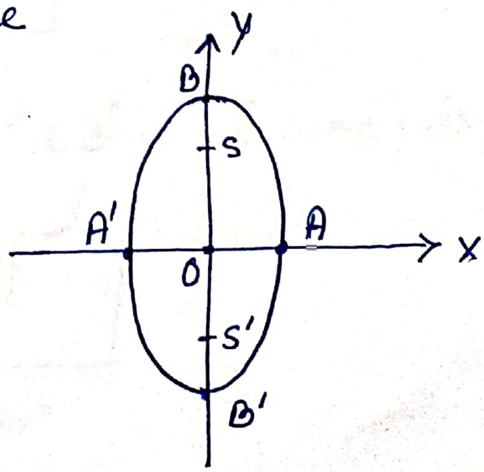
(v) Foci are $(\pm ae, 0)$

(vi) Directrices are $x = \pm \frac{a}{e}$.

(vi) Length of latus rectum $= \frac{2b^2}{a}$.

(B) Standard equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \text{ where } b > a$$



(i) Centre is $(0,0)$

(ii) Eccentricity e is given by

$$a^2 = b^2(1-e^2)$$

(iii) The major axis is along the Y-axis & the length of major axis $= 2b$.

(iv) The minor axis is along the X-axis & the length minor axis $= 2a$.

(v) Foci are $(0, \pm be)$

(vi) Directrices are $y = \pm \frac{b}{e}$

(vii) Length of Latus rectum $= \frac{2a^2}{b}$.

Equation of tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\boxed{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$

Equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\boxed{\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}}$

or $\boxed{\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 - b^2}$

The line $y = mx + c$ will be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $\boxed{c^2 = a^2m^2 + b^2}$

i.e. $\boxed{c = \pm\sqrt{a^2m^2 + b^2}}$

The equation of any tangent to the ellipse is $y = mx + \sqrt{a^2m^2 + b^2}$.

It point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$.

<u>Parametric equation of an ellipse</u>

The parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by

$$\boxed{x = a\cos\theta \quad , \quad y = b\sin\theta.}$$

The equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$$\boxed{\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1}$$

The equation of normal at $(a\cos\theta, b\sin\theta)$ is

$$\boxed{ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2}$$

Auxiliary circle:

The circle described on the major axis of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ as diameter is called auxiliary circle of the ellipse. Its equation is given by $x^2+y^2=a^2$.

Examples:

① Prove that the $lx+my+n=0$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $a^2l^2+b^2m^2=n^2$.

Sol^n: We know that the line $y=m'x+c$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$c^2=a^2m'^2+b^2 \quad \text{——} ①$$

Given, eq^n of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ &

eq^n of line is $lx+my+n=0$

$$\Rightarrow my=-lx-n$$

$$\Rightarrow y=\left(\frac{-l}{m}\right)x+\left(\frac{-n}{m}\right)$$

Comparing with $y=m'x+c$, we get

$$m'=\frac{-l}{m}, \quad c=\frac{-n}{m}.$$

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Putting these values in 1, we get

$$c^2 = a^2 m'^2 + b^2$$

$$\Rightarrow \left(\frac{-n}{m}\right)^2 = a^2\left(\frac{-l}{m}\right)^2 + b^2$$

$$\Rightarrow \frac{n^2}{m^2} = \frac{a^2 l^2}{m^2} + b^2$$

$$\Rightarrow \frac{n^2}{m^2} = \frac{a^2 l^2 + b^2 m^2}{m^2}$$

$$\Rightarrow a^2 l^2 + b^2 m^2 = n^2$$

Proved

Q. Prove that the line $lx + my = n$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$.

Soln: Suppose the line $lx + my = n$ ——① is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.

Then the eqn of the normal is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2 \quad \text{——②}$$

Comparing ① & ②, we get

$$\frac{a\sec\theta}{l} = \frac{-b\operatorname{cosec}\theta}{m} = \frac{a^2-b^2}{n}$$

$\therefore \frac{a\sec\theta}{l} = \frac{a^2-b^2}{n}$, $\frac{-b\operatorname{cosec}\theta}{m} = \frac{a^2-b^2}{n}$

$\Rightarrow \frac{a}{l\cos\theta} = \frac{a^2-b^2}{n}$ $\qquad \Rightarrow \frac{-b}{m\sin\theta} = \frac{a^2-b^2}{n}$

$\Rightarrow \frac{a}{l} = \left(\frac{a^2-b^2}{n}\right)\cos\theta$ ——③ $\qquad \Rightarrow \frac{-b}{m} = \left(\frac{a^2-b^2}{n}\right)\sin\theta$ ——④

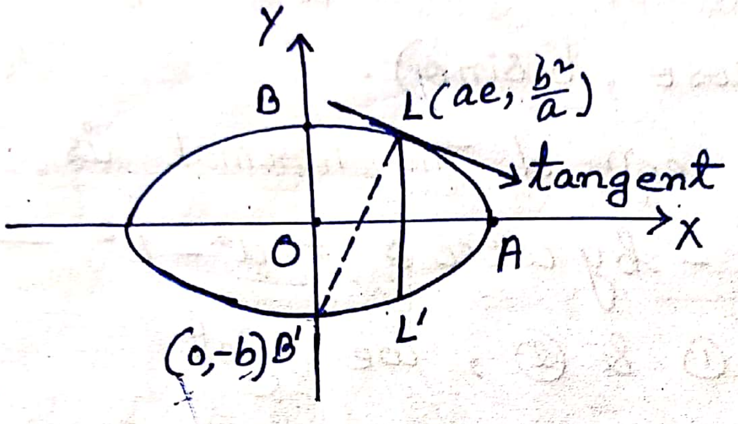
Now, $③^2 + ④^2 \Rightarrow \frac{a^2}{l^2} + \frac{b^2}{m^2} = \left(\frac{a^2-b^2}{n}\right)^2(\cos^2\theta + \sin^2\theta)$

$$= \frac{(a^2-b^2)^2}{n^2}.$$

Proved

Q. If the normal at an end of a latus rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the curve is given by the equation $e^4+e^2-1=0$.

Solⁿ:



The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ — (1)

The co-ordinates of the extremity L of the latus rectum LL' of the ellipse (1) are $\left(ae, \frac{b^2}{a}\right)$.

The eqⁿ of the normal at the point $L\left(ae, \frac{b^2}{a}\right)$ is $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}$.

$$\Rightarrow \frac{x-ae}{\frac{ae}{a^2}}=\frac{y-\frac{b^2}{a}}{\frac{b^2}{ab^2}}$$

$\because$ The normal passes through the extremity $(0,-b)$ of the minor axis.

$$\therefore \frac{0 - ae}{\frac{ae}{a^2}} = \frac{-b - \frac{b^2}{a}}{\frac{1}{a}}$$

$$\Rightarrow -a^2 = -ab - b^2$$

$$\Rightarrow a^2 = ab + b^2$$

$$\Rightarrow a^2 = a \cdot a\sqrt{1-e^2} + a^2(1-e^2) \quad \left[\because b^2 = a^2(1-e^2)\right]$$

$$\Rightarrow 1 = \sqrt{1-e^2} + (1-e^2)$$

$$\Rightarrow e^2 = \sqrt{1-e^2}$$

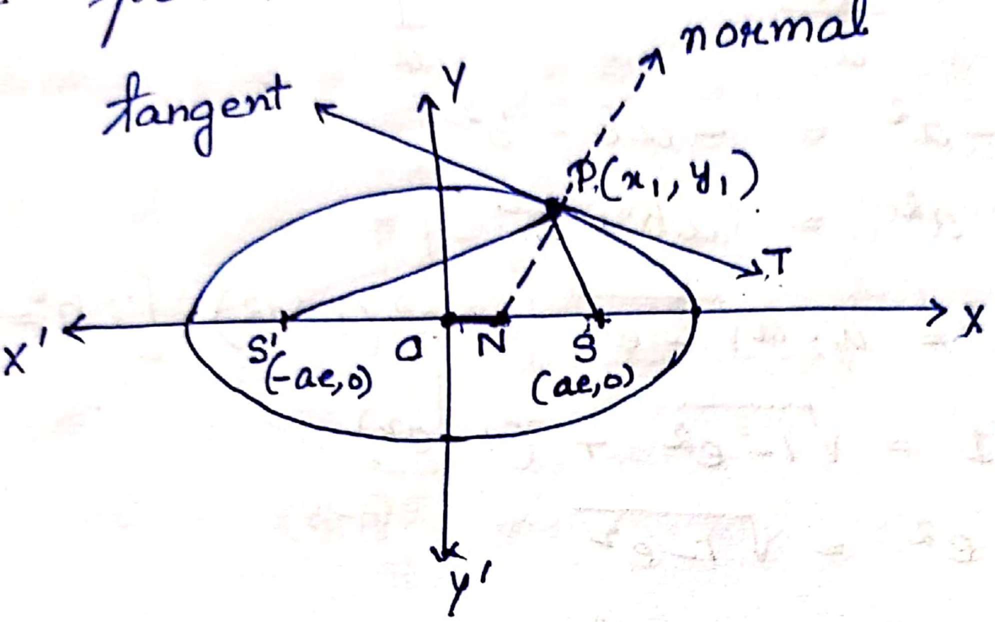
$$\Rightarrow e^4 = 1 - e^2$$

$$\Rightarrow e^4 + e^2 - 1 = 0 \quad //$$

Proved

Ex: Prove that the tangent & normal at any point of an ellipse bisect the external & internal angles between the focal radii of that point.

Sol^n:



Let the eq^n of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —— (1)

Let $P(x_1, y_1)$ be any point on the ellipse.

The eq^n of the normal at P is

$$\frac{a^2(x-x_1)}{x_1} = \frac{b^2(y-y_1)}{y_1} \quad \text{—— (2)}$$

Let the normal (2) meet the major axis at N.

Then the x-coordinate of N will be given by putting $y=0$ in (2).

$\therefore$ ② $\Rightarrow \quad \dfrac{a^2(x-x_1)}{x_1} = b^2\left(\dfrac{0-y_1}{y_1}\right)$

$\Rightarrow \quad \dfrac{a^2(x-x_1)}{x_1} = b^2(-1)$

$\Rightarrow \quad a^2(x-x_1) = -b^2 x_1$

$= -a^2(1-e^2)x_1 \quad [\because b^2 = a^2(1-e^2)]$

$\Rightarrow a^2x - a^2x_1 = -a^2x_1 + a^2e^2x_1$

$\Rightarrow \quad a^2x = a^2e^2x_1$

$\Rightarrow \quad x = e^2x_1$

$\therefore \quad ON = \cancel{a} \cdot e^2x_1$

Now, $SN = OS - ON = ae - \cancel{a}\, e^2x_1$

$= e(a - ex_1)$

$= e\,SP$ [by focal property)

& $S'N = OS' + ON$

$= ae + e^2x_1$

$= e(a + ex_1)$

$= e\,S'P$ [by focal property]

$\therefore \dfrac{SN}{S'N} = \dfrac{eSP}{eS'P} = \dfrac{SP}{S'P}$

$\therefore$ Normal PN bisect $\angle S'PS$ between the focal radii of the point P, in the Δ PSS'.

Again, we know that the tangent PT is perpendicular to the normal PN & therefore the tangent PT is the external bisector of the angle $\angle SPS'$.

Q. If the chord through the points whose eccentric angles are α & β on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, passes through a focus, prove that $\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} + \left\{\frac{1-e}{1+e}\right\} = 0$.

Sol^n: We know, the eq.^{n} of the chord joining the points α & β is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\cdot\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right) \quad \text{—(1)}$$

$\because$ This chord passes through the focus $(ae, 0)$, we have

$$\frac{ae}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + 0 = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{e}{1}$$

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$$\Rightarrow \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1}{e}$$

Applying componendo & dividendo, we have

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{1-e}{1+e}$$

$$\Rightarrow \frac{-2.\sin\left(\frac{\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}}{2}\right).\sin\left(\frac{\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}}{2}\right)}{2.\cos\left(\frac{\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}}{2}\right).\cos\left(\frac{\frac{\alpha+\beta}{2}-\frac{(\alpha-\beta)}{2}}{2}\right)} = \frac{1-e}{1+e}$$

$$\Rightarrow \frac{-2.\sin\left(\frac{\alpha}{2}\right).\sin\left(\frac{\beta}{2}\right)}{2.\cos\left(\frac{\alpha}{2}\right).\cos\left(\frac{\beta}{2}\right)} = \frac{1-e}{1+e}$$

$$\Rightarrow -\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{1-e}{1+e}$$

$$\Rightarrow 0 = \tan\left(\frac{\alpha}{2}\right).\tan\left(\frac{\beta}{2}\right) - \frac{1-e}{1+e}$$

Proved

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Ex: Show that the line $lx + my + n = 0$ will cut the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\frac{\pi}{2}$ if $a^2l^2 + b^2m^2 = 2n^2$.

Sol^n The eq^n of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let the line $lx + my + n = 0$ (1) cuts the ellipse in points P & Q whose eccentric angles are α & β respectively, such that

$$\alpha - \beta = \frac{\pi}{2} \quad \Rightarrow \alpha = \frac{\pi}{2} + \beta \quad \text{--- (2)}$$

The eq^n of PQ is

$$\frac{x}{a} \cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b} \cdot \sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \frac{x}{a} \cos \frac{1}{2}\left(\frac{\pi}{2} + \beta + \beta\right) + \frac{y}{b} \sin \frac{1}{2}\left(\frac{\pi}{2} + \beta + \beta\right)$$
$$= \cos \frac{1}{2}\left(\frac{\pi}{2} + \beta - \beta\right) \quad [\text{Using (2)}]$$

$$\Rightarrow \frac{x}{a} \cos \frac{1}{2}\left(\frac{\pi}{2} + 2\beta\right) + \frac{y}{b} \sin \frac{1}{2}\left(\frac{\pi}{2} + 2\beta\right) = \cos \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{x}{a} \cos\left(\frac{\pi}{4} + \beta\right) + \frac{y}{b} \sin\left(\frac{\pi}{4} + \beta\right) = \cos \frac{\pi}{4} \quad \text{--- (3)}$$

Comparing (1) & (3), we get

$$\frac{\frac{\cos\left(\frac{\pi}{4} + \beta\right)}{a}}{l} = \frac{\frac{\sin\left(\frac{\pi}{4} + \beta\right)}{b}}{m} = \frac{\cos \frac{\pi}{4}}{-n}$$

$$\Rightarrow \frac{\cos\left(\frac{\pi}{4}+\beta\right)}{al} = \frac{\sin\left(\frac{\pi}{4}+\beta\right)}{bm} = \frac{\cos\left(\frac{\pi}{4}\right)}{-n}$$

$$\Rightarrow \frac{\cos\left(\frac{\pi}{4}+\beta\right)}{al} = \frac{\cos\left(\frac{\pi}{4}\right)}{-n}, \quad \frac{\sin\left(\frac{\pi}{4}+\beta\right)}{bm} = \frac{\cos\left(\frac{\pi}{4}\right)}{-n}$$

$$\Rightarrow \cos\left(\frac{\pi}{4}+\beta\right) = \frac{al}{-\sqrt{2}\,n} \quad \text{--- (4)} \qquad \Rightarrow \sin\left(\frac{\pi}{4}+\beta\right) = \frac{bm}{-\sqrt{2}\,n} \quad \text{--- (5)}$$

$$(4)^2 + (5)^2 \Rightarrow \left(\frac{al}{-\sqrt{2}n}\right)^2 + \left(\frac{bm}{-\sqrt{2}n}\right)^2 = \cos^2\left(\frac{\pi}{4}+\beta\right) + \sin^2\left(\frac{\pi}{4}+\beta\right)$$

$$\Rightarrow \frac{a^2 l^2}{2n^2} + \frac{b^2 m^2}{2n^2} = 1$$

$$\Rightarrow a^2 l^2 + b^2 m^2 = 2n^2$$ Proved