Director Circle

The director circle of an ellipse is the locus of a point where two perpendicular tangent can be drawn to the ellipse.

Ex: Find the equation of the director circle.

Soln Let the eq.n of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$$

We know the eq.n of any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y - mx = \sqrt{a^2m^2 + b^2} \quad \text{--- (2)}$$

The eq.n (2) is a tangent to the ellipse (1) for all values of m.

The eq.n of a tangent perpendicular to the tangent (2) is

$$y - \left(\frac{-1}{m}\right)x = \sqrt{a^2\left(-\frac{1}{m}\right)^2 + b^2} \quad \left[\text{Put } m = -\frac{1}{m} \text{ in (2)}\right]$$

$$\Rightarrow \frac{my + x}{m} = \frac{\sqrt{a^2 + m^2 b^2}}{m}$$

$$\Rightarrow my + x = \sqrt{a^2 + m^2 b^2} \quad \text{--- (3)}$$

Now, $(2)^2 + (3)^2 \Rightarrow (y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2 + m^2b^2$

$$\Rightarrow y^2 + m^2x^2 - 2ymx + m^2y^2 + x^2 + 2myx = a^2m^2 + a^2 + b^2 + b^2m^2$$

$$\Rightarrow y^2(1+m^2) + x^2(1+m^2) = a^2(1+m^2) + b^2(1+m^2)$$

$$\Rightarrow y^2 + x^2 = a^2 + b^2$$

$$\Rightarrow x^2 + y^2 = a^2 + b^2$$

This is the required eq.n of the director circle.//

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Diameter

Any chord of an ellipse passing through the centre of the ellipse is called a diameter of the ellipse.

Ex: Find the locus of the middle points of a system of parallel chords of an ellipse.

Sol.^{n} Let the eq.^{n} of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ — (1)

Let the eq.^{n} of a system of parallel chords of the ellipse be

$y = mx + c$, — (2) where c is a parameter & m is a constant.

Let (h, k) be the middle point of the chord.

Then $k = mh + c$ — (3)

From (1) & (2), we have

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow \frac{b^2x^2 + a^2(m^2x^2 + c^2 + 2mxc)}{a^2b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2m^2x^2 + a^2c^2 + 2a^2mcx = a^2b^2$$

$$\Rightarrow a^2m^2x^2 + b^2x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow (a^2m^2+b^2)x^2 + 2a^2mc\,x + a^2(c^2-b^2) = 0 \quad \text{--- (4)}$$

which is a quadratic equation in x.

Let x_1 & x_2 be the roots of the eq^n (4)

Then $x_1 + x_2 = \dfrac{-2a^2mc}{a^2m^2+b^2}$

$$\therefore h = \frac{x_1+x_2}{2} = \frac{-2a^2mc}{2(a^2m^2+b^2)}$$

$$\Rightarrow (a^2m^2+b^2)h = -a^2mc$$

$$\Rightarrow c = \frac{-h(a^2m^2+b^2)}{a^2m}$$

Putting this value in (3), we get

$$(3) \Rightarrow k = mh + c$$

$$\Rightarrow k = mh - \frac{h(a^2m^2+b^2)}{a^2m}$$

$$= \frac{a^2m^2h - a^2m^2h - b^2h}{a^2m}$$

$$\Rightarrow k = \frac{-hb^2}{a^2m}$$

$\therefore$ The locus of the middle point is

$$y = \frac{-b^2}{a^2m}x$$

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Conjugate diameters :

Two diameters are called conjugate diameters when each bisects chords parallel to the other.

The two diameters $y = m_1 x$ & $y = m_2 x$ are conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{if} \quad m_1 m_2 = \frac{-b^2}{a^2}.$$

Ex: Show that the sum of the reciprocals of the squares of any two diameters of an ellipse which are at right angles to one another is constant.

Soln: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ —(1)

Let the eqns of two perpendicular diameters of (1) be $y = mx$ —(2) & $y = \frac{-1}{m}x$ —(3)

Putting $y = mx$ in eqn (1), we get

$$\frac{x^2}{a^2} + \frac{(mx)^2}{b^2} = 1$$

$$\Rightarrow x^2\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) = 1$$

$$\Rightarrow x^2\left(\frac{b^2 + a^2m^2}{a^2b^2}\right) = 1.$$

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$$\Rightarrow x^2 = \frac{a^2 b^2}{b^2 + m^2 a^2} \quad \text{——— (4)}$$

Now, Eq^n (2) $\Rightarrow$

$$y = mx$$

$$\Rightarrow y^2 = m^2 x^2$$

$$= m^2 \cdot \frac{a^2 b^2}{b^2 + m^2 a^2} \quad [\text{Using (4)}]$$

$$\text{——— (5)}$$

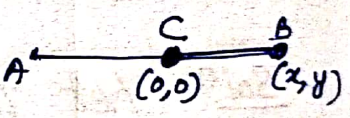
Now, (4) + (5) $\Rightarrow$

$$x^2 + y^2 = \frac{a^2 b^2}{b^2 + m^2 a^2} + \frac{m^2 a^2 b^2}{b^2 + m^2 a^2}$$

$$= \frac{a^2 b^2}{b^2 + m^2 a^2}(1 + m^2)$$

$\therefore$ The length of the semi-diameter given by $y = mx$

= distance between the centre $(0,0)$ and the point (x, y)



$$= \sqrt{(x-0)^2 + (y-0)^2}$$

$$= \sqrt{x^2 + y^2}$$

$$= \sqrt{\frac{a^2 b^2}{b^2 + m^2 a^2}(1 + m^2)}$$

$$= ab \sqrt{\frac{1 + m^2}{b^2 + m^2 a^2}}$$

$\therefore$ l_1 = length of the diameter $y = mx$

$$= 2ab \sqrt{\frac{1 + m^2}{b^2 + m^2 a^2}}$$

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Then $\frac{1}{l_1} = \frac{1}{2ab}\sqrt{\frac{b^2+m^2a^2}{1+m^2}}$ $y = mx$

$$\Rightarrow \frac{1}{l_1^2} = \frac{b^2+m^2a^2}{4a^2b^2(1+m^2)} \quad \text{—— (6)}$$

Let l_2 = length of the diameter $y = (-\frac{1}{m})x$

Putting $m = -\frac{1}{m}$ in (6), we get

$$\frac{1}{l_2^2} = \frac{b^2 + (-\frac{1}{m})^2 a^2}{4a^2b^2(1+(-\frac{1}{m})^2)} = \frac{b^2 + \frac{a^2}{m^2}}{4a^2b^2(\frac{m^2+1}{m^2})}$$

$$= \frac{b^2m^2+a^2}{4a^2b^2(m^2+1)} \quad \text{—— (7)}$$

Now, $\frac{1}{l_1^2} + \frac{1}{l_2^2} = \frac{b^2+m^2a^2}{4a^2b^2(1+m^2)} + \frac{a^2+b^2m^2}{4a^2b^2(1+m^2)}$

$$= \frac{(a^2+b^2) + m^2(a^2+b^2)}{4a^2b^2(1+m^2)}$$

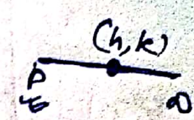
$$= \frac{(a^2+b^2)(1+m^2)}{4a^2b^2(1+m^2)}$$

$$\Rightarrow \frac{1}{l_1^2} + \frac{1}{l_2^2} = \frac{a^2+b^2}{4a^2b^2}$$

Proved. //

Ex: Prove that if CP & CD be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the locus of the middle point of PD is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}.$$



Sol^n Given equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

and CP & CD are two conjugate semi-diameters.

Let coordinate of P be $(a\cos\theta, b\sin\theta)$.

Then co-ordinate of D is $(-a\sin\theta, b\cos\theta)$

Let (h, k) be the middle point of PD.

$$\boxed{\begin{aligned} h &= \frac{x_1+x_2}{2} \\ k &= \frac{y_1+y_2}{2} \end{aligned}}$$

Then, $h = \frac{a\cos\theta + (-a\sin\theta)}{2}$

$$\Rightarrow \frac{2h}{a} = \cos\theta - \sin\theta \quad \text{——} (1)$$

& $k = \frac{b\sin\theta + b\cos\theta}{2}$

$$\Rightarrow \frac{2k}{b} = \sin\theta + \cos\theta. \quad \text{——} (2)$$

$$(1)^2 + (2)^2 \Rightarrow \left(\frac{2h}{a}\right)^2 + \left(\frac{2k}{b}\right)^2 = (\cos\theta - \sin\theta)^2 + (\sin\theta + \cos\theta)^2$$

$$\Rightarrow \frac{4h^2}{a^2} + \frac{4k^2}{b^2} = \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta -$$
$$+ \sin^2\theta + \cos^2\theta + 2\cos\theta\sin\theta$$

$$\Rightarrow 4\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right) = 1+1$$
$$= 2$$

$$\Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2} = \frac{2}{4} = \frac{1}{2}$$

$\therefore$ Locus of the middle point (h, k) of PQ

is $$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{1}{2}$$

<u>Proved</u>