Employment of only row transformation

Theorem: If A be a $m \times n$ matrix of rank r then $\exists$ a non-singular matrix P such that

$$PA = \begin{bmatrix} G \\ O \end{bmatrix}$$

where, G is an $r \times n$ matrix of rank r & O zero is $(m-r) \times n$.

Proof: Since A is a $m \times n$ matrix of rank r therefore $\exists$ non-singular matrices P & Q ~~st~~ such that

$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \quad \text{———} \ (1)$$

Now, every non-singular matrix can be expressed as a product of elementary matrices.

Let, ~~Q is~~ $Q = Q_1 Q_2 \cdots Q_t$

where $Q_1, Q_2, \cdots, Q_t$ are all elementary matrices.

Thus, the relation (1) can be written as

$$PAQ_1 Q_2 \cdots Q_t = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \quad \text{———} \ (2)$$

Now, every column tranformation of a matrix is equivalent to post multiplication with the correspondg E-matrix. Since no column transformation can affect the last

$(m-r)$ rows of the write hand side of equ^n ②
Therefore, by post multiplying the LHS of equ^n ② by the E - matrices ~~QT^{-1}, QT~~ ~~$Q_t^{-1}, Q_{t-1}^{-1} \cdots Q_2^{-1}$~~

$Q_t^{-1} Q_{t-1}^{-1} \cdots Q_2^{-1} Q_1^{-1}$ successively & effecting the corresponding column transformation of the RHS of equ^n ② we get

$$PA = \begin{bmatrix} G \\ O \end{bmatrix}$$

. Since E-transformation doesnot alter the rank therefore the rank of the matrix PA is the same as that of the matrix A which is r.

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Employment of only column transformation

Theorem : If A be a $m \times n$ matrix of rank r then $\exists$ a non-singular matrix Q such that

$$AQ = [H \quad O]$$

where, H is an $m \times r$ matrix of rank r & O is an $m \times (n-r)$

Proof : Since A is an $m \times n$ matrix of rank r, therefore $\exists$ non-singular matrices P & Q such that

$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \quad \text{———} ①$$

Now, every non-singular matrix can be expressed as the product of elementary matrices. So, let

$P = P_1 P_2 \cdots P_s$, where $P_1, P_2, \cdots, P_s$ are elementary matrices.

Thus, the relation ① can be written as

$$P_1 P_2 \cdots P_s AQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{——— ②}$$

Now, every E-row transformation of a matrix is equivalent to pre-multiplication with the corresponding elementary matrix. Again no row transformation can affect the last $n-r$ columns of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Therefore, pre-multiplying the L.H.S. of ⑪ by the elementary matrices $P_1^{-1}, P_2^{-1}, \cdots, P_s^{-1}$ successively and effecting the corresponding transformations of R.H.S. of ⑪ we get a relation of the form $AQ = [H \quad 0]$.

Now elementary transformations do not alter the rank. Therefore the rank of the matrix AQ is the same as that of A which is r. Thus the rank of the matrix $[H \quad 0]$ is r and therefore the rank of the matrix H is also r as the matrix H has r columns and the last $n-r$ columns of the matrix $[H \quad 0]$ consist of zeros only.

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The rank of a product :

Theorem : The rank of a product of two matrices can not exceed the rank of either matrix.

Proof : Let, A & B be two $m \times n$ & $n \times p$ matrices respectively. Let, r_1, r_2 be the rank of A & B. respectively & let, r be the rank of the product AB.

To prove that $r \leq r_1$ & $r \leq r_2$.

Since A is a $m \times n$ matrix of rank r_1 therefore $\exists$ a a non-singular matrix P such that

$$PA = \begin{bmatrix} G \\ O \end{bmatrix}$$

where G is an $r_1 \times n$ matrix of rank r & O is $(m - r_1) \times n$ ~~thereof the~~

$$\therefore PAB = \begin{bmatrix} G \\ O \end{bmatrix} B$$

Since, the rank of a matrix does not alter by multiplying with a non-singular matrix

$$\therefore \text{Rank}(PAB) = \text{Rank}(AB) = r.$$

$\therefore$ Rank of the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B = r$

Since the matrix G has only r_1 non-zero rows therefore the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B$ can not have more than r_1 non zero rows. which arise by multiplying the r_1 non zero rows of B with the columns of B.

$\therefore$ Rank of the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B$ is $\leq r_1$

i.e, $\rho(AB) \leq$ rank of the prefactor A

$$\rho(AB) \leq \rho(A)$$

Again, $r = \rho(AB) = \rho(\overline{AB})' = \rho(B'A') \leq \rho(B')$
$= \rho(B) = r_2$

i.e, $r \leq r_2$.

i.e, $\rho(AB) \leq$ ~~$\rho(B)$~~ rank of the post factor B.

Hence proved.

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Assignment

Topic: Invariance of row rank under E-row transformation.

column

Theorem

① Row equivalent matrices have the same row rank. 201

② Row equivalent matrices have the same column rank. 202

③ If r, be the row rank of an $m \times n$ matrix A then there exists a non-singular matrix, P such that

$$PA = \begin{bmatrix} K \\ O \end{bmatrix}$$

where, k is an $r \times n$ matrix consisting of a set of r linearly independent rows of A. 203

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Rank of a sum :

Theorem : Rank of the sum of two matrices cannot exceed the sum of their rank.

Proof : Let, A & B be two matrices of the same type. Let, S_A, S_B, S_{A+B} denote the row spaces of the matrices A, B, $A+B$ respectively. Let, S denote the sub space generated jointly by the rows of A as well as the rows of B.

Now, the no. of members in a basis of S must be less than or equal to the sum of the nos of members in the basis of A & B.

$\therefore$ Dimension $S \leq$ Dimension S_A + Dimension S_B

Again, the row spaces S_{A+B} is a sub space of S

$\therefore$ Dimension $S_{A+B} \leq$ Dimension S

$\therefore$ Dimension $S_{A+B} \leq$ Dimension S_A + Dimension S_B

$\therefore$ Row rank $(A+B) \leq$ Row rank (A) + Row rank (B)

$\Rightarrow$ Rank $(A+B) \leq$ rank (A) + rank (B) [$\because$ Row rank and rank of a matrix are equal.]

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Theorem: If A, B are two n rowed square matrices then $\text{Rank}(AB) \geq \text{Rank } A + \text{Rank } B - n$ ✓

Proof: Let, r be the rank of matrix A then $\exists$ two non-singular matrix P & Q $\ni$

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{———} (1)$$

Now, premultiplying both sides of (1) by P^{-1} we get—

$$\Rightarrow PP^{-1}AQ = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow IAQ = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow AQ = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{———} (2)$$

Postmultiplying both sides of (2) by Q^{-1} we get—

$$AQQ^{-1} = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

$$\Rightarrow AI = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

$$\Rightarrow A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Let us now consider another matrix C

$$C = P^{-1}\begin{bmatrix} O_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Then, $A+C = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} + P^{-1}\begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$

$$= P^{-1} \begin{bmatrix} I_r & O \\ O & I_{n-r} \end{bmatrix} Q^{-1}$$

$$= P^{-1} I_n Q^{-1}$$

$$= P^{-1} Q^{-1}$$

Thus, $A+C$ is a non-singular matrix of order n.

$\therefore$ Rank of $(A+C) = n$

$\therefore$ Rank $C = n - r$

$= n -$ rank A

Now, Rank $\{(A+C)B\}$ = Rank B, as the matrix $A+C$ is a non singular and the rank of a matrix is not altered by premultiplication with a non-singular matrix.

Thus, Rank $B = [(A+C)B]$ = Rank $(AB + CB)$

$\therefore$ Rank $B \leq$ Rank (AB) + Rank (CB) —— (3)

Again, Rank$(CB) \leq$ Rank C [$\because$ The rank of the product of two matrices is less than or equal to the rank of either matrix.]

$\therefore$ Rank $(CB) \leq n -$ Rank A —— ~~(3)~~ (4)

From (3) & (4) we get

Rank $B \leq$ Rank $(AB) + n -$ Rank(A)

$\Rightarrow$ Rank $(AB) \geq$ Rank A + Rank $(B) - n$

Hence proved//

Example: If A be any non-singular matrix & B be a matrix $\ni AB$ exists, then show that AB & B have the same rank.

Soln: Let, $C = AB$

Since A is an non singular matrix

$$\therefore B = A^{-1}C$$

Now, we know that the rank of the product of two matrices can not exceed the rank of either matrix.

$$\therefore \text{Rank } C = \text{Rank}(AB) \leq \text{Rank}(B)$$

$$\Rightarrow \text{Rank}(B) = \text{Rank}(AA^{-1}C) = \text{Rank}(C)$$

$$\Rightarrow \text{Rank}(B) \geq \text{Rank } C = \text{Rank}(AB)$$

$$\Rightarrow \text{Rank}(B) = \text{Rank } C = \text{Rank}(AB)$$

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Q Show that if the ~~m~~ $n \times n$ matrix A satisfies the equn $A^2 = A$. then, rank A + rank $(I - A) = n$

Soln: A is an $n \times n$ matrix $\ni$

$$A^2 - A = 0$$

$$\Rightarrow A(A - I_n) = 0$$

$$\Rightarrow A(I_n - A) = 0$$

Now; the sum of the matrices A & $(I_n - A)$ is the matrix I_n & we know that the rank of the sum of two matrices can not exiced sum of ~~the~~ there rank.

$$\therefore \text{Rank}[A + (I_n - A)] \leq \text{Rank of } A + \text{Rank}(I_n - A)$$

$\Rightarrow \text{Rank}(I_n) \leqslant \text{Rank } A + \text{Rank}(I_n - A)$

$\Rightarrow n \leqslant \text{Rank } A + \text{Rank of } (I_n - A)$

$\therefore \text{Rank}(A) + \text{Rank}(I_n - A) = n$ proved.//

Q. If A be an $n \times n$ matrix, show that the rank of adj A is n, 1 or 0 according as the rank of A is n, $n-1$ or less than $n-1$.

Solⁿ: (i) Let A be an $n \times n$ matrix.

Then $A(\text{adj } A) = |A| I_n$

$\therefore |A||\text{adj } A| = |A||I_n| = A$

Since the matrix A is of rank n, therefore $|A| \neq 0$.

$\therefore |A||\text{adj } A| = |A|$ gives $|\text{adj } A| = 1$.

Thus, the matrix adj A is also non-singular. Hence, it is of rank n.

(ii) If the rank of A is $n-1$, then at least one minor of order $n-1$ of the matrix A is not equal to zero. Therefore the matrix adj A will be a non-zero matrix & thus the rank of the matrix adj A will be greater than zero.

Again the rank of the matrix A is $n-1$. Therefore $|A| = 0$. Therefore $A(\text{adj } A)$ is a zero matrix and therefore is of rank zero. Hence, we have

$$0 \geqslant \text{Rank } A + \text{Rank adj } A - n$$

$\Rightarrow$ Rank A + Rank adj $A \leq n$

$\Rightarrow n-1$ + Rank adj $A \leq n$

$\Rightarrow$ Rank adj $A \leq 1$

But we have shown that Rank adj $A > 0$

Hence Rank adj $A = 1$

(iii) If the rank of A is less than $n-1$, then all minors of order $n-1$ of the matrix A will be zero. Therefore the matrix adj A will be a zero matrix & hence Rank adj A will be zero.