Quadratic forms

The rank of a quadratic form

Let $X'AX$ be a quadratic form over a field F. The rank of a matrix A is called the rank of the quadratic form $X'AX$.

If $X'AX$ is a quadratic form of rank r then $\exists$ a non singular matrix P which will reduce the form $X'AX$ to a sum of r sq. terms.

Working rule for numerical problems

We should transform the given symmetric matrix A to a diagonal form by applying conguent operations then the application of column operations to the ~~unique~~ unit matrix I_n will give us a non singular matrix P such that

$$P'AP = \text{a diagonal matrix}$$

Congruence of matrices

A sq. matrix B of order n over a field F is said to be congruence to another sq. matrix A of order n over F if $\exists$ a non-singular matrix P over F $\ni$

$$B = P'AP$$

**
Define quadratic form.

An expression of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

where a_{ij}'s are elements of a field F is called a quadratic form n variables $x_1 x_2 \cdots x_n$ over a field F.

Matrix of a quadratic form

If $\phi = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ is a quadratic form in n variables $x_1 x_2 \cdots x_n$ then $\exists$ a unique symmetric matrix B of order n $\ni \phi = X'BX$ where $X = [x_1\ x_2 \cdots x_n]'$.
The symmetric matrix B is called the matrix of the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Q. Write down the matrix of each of the following quadratic forms & varify they can be written as a matrix product $X'AX$.

Soln: (i) $x_1^2 - 18x_1x_2 + 5x_2^2$

Soln: The given quadratic form can be written as

$$x_1x_1 - 9x_1 - 9x_1 + 5x_2x_2$$

Let, A be the matrix of the quadratic form

$$A = \begin{pmatrix} 1 & -9 \\ -9 & 5 \end{pmatrix}$$

Let, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $X' = (x_1, x_2)$

$$X'A = (x_1, x_2)\begin{pmatrix} 1 & -9 \\ -9 & 5 \end{pmatrix}$$

$$= (x_1 - 9x_2, -9x_1 + 5x_2)$$

$$\therefore X'AX = (x_1 - 9x_2, -9x_1 + 5x_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\cancel{= x_1^2 - 9x_1x_2, -9x_1x_2 + 5x_1x_2}$$

$$= (x_1^2 - 9x_1^2x_2 + (-9x_1x_2) + 5x_2^2)$$

$$= x_1^2 - 18x_1x_2 + 5x_2^2$$

(ii) $x_1^2 + 2x_2^2 - 5x_3^2 + 4x_2x_3 - 3x_3x_1 - x_1x_2$

Sol^n: The given ~~qu~~ quadratic form can be written as

$$x_1x_1 + 2x_2x_2 - 5x_3x_3 + 2x_2x_3 + 2x_2x_3 - \tfrac{3}{2}x_3x_1 - \tfrac{3}{2}x_3x_1 - \tfrac{1}{2}x_1x_2 - \tfrac{1}{2}x_1x_2$$

Let A be the matrix of the quadratic form

$$A = \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\tfrac{1}{2} & -\tfrac{3}{2} \\ -\tfrac{1}{2} & 2 & 2 \\ -\tfrac{3}{2} & 2 & -5 \end{pmatrix}$$

Let, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $X' = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$

$$X'A = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & -\tfrac{1}{2} & -\tfrac{3}{2} \\ -\tfrac{1}{2} & 2 & 2 \\ -\tfrac{3}{2} & 2 & -5 \end{pmatrix}$$

$$= \left(x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3,\ -\tfrac{1}{2}x_1 + 2x_2 + 2x_3,\ -\tfrac{3}{2}x_1 + 2x_2 - 5x_3\right)$$

$$\cancel{X'AX = \left(x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3,\ -\tfrac{1}{2}x_1 + 2x_2 + 3x_3,\ -\tfrac{3}{2}x_1 + 2x_2 - 5x_3\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$$

$$\cancel{= x_1^2 - \tfrac{1}{2}x_1x_2 - \tfrac{3}{2}x_1x_3,\ -\tfrac{1}{2}x_1^2 + 2x_1x_2 + 3x_1x_3,\ -\tfrac{3}{2}x_1^2 + 2x_1x_2 - 5x_1x_3}$$

$AX = \left(x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3,\ -\tfrac{1}{2}x_1 + 2x_2 + 2x_3,\ -\tfrac{3}{2}x_1 + 2x_2 - 5x_3\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$= \begin{pmatrix} x_1^2 - \tfrac{1}{2}x_1x_2 - \tfrac{3}{2}x_1x_3, & -\tfrac{1}{2}x_1^2 + 2x_1x_2 + 3x_1x_3, & -\tfrac{3}{2}x_1^2 + 2x_1x_2 - 5x_1x_3 \\ x_1x_2 - \tfrac{1}{2}x_2^2 - \tfrac{3}{2}x_2x_3, & -\tfrac{1}{2}x_1x_2 + 2x_2^2 + 3x_2x_3, & -\tfrac{3}{2}x_1x_2 + 2x_2^2 - 5x_2x_3 \\ x_1x_3 - \tfrac{1}{2}x_2x_3 - \tfrac{3}{2}x_3^2, & -\tfrac{1}{2}x_1x_3 + 2x_2x_3 + 3x_3^2, & -\tfrac{3}{2}x_1x_3 + 2x_2x_3 - 5x_3^2 \end{pmatrix}$$

$$= x_1^2 - \tfrac{1}{2}x_1x_2 - \tfrac{3}{2}x_1x_3 - \tfrac{1}{2}x_1x_2 + 2x_2^2 + 3x_2x_3 - \tfrac{3}{2}x_1x_3 + 2x_2x_3 - 5x_3^3$$

$$\Rightarrow x_1^2 + 2x_2^2 - 5x_3^3 - x_1x_2 - 3x_1x_3 + 5x_2x_3$$

9/10/19

Corollary

corresponding to every quadratic form $X'AX$ over a field F, $\exists$ a non-~~linear~~ singular linear transformation

$X = PY$, over F, $\ni$ the form $X'AX$ transform to a sum of r, sq. terms

9/10/19 $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_r y_r^2$ where $\lambda_1, \lambda_2, \ldots, \lambda_r$ belong to the field F & r is the rank of the matrix A.

Q. Define rank of a quadratic form.

Ans $\Rightarrow$ Let $X'AX$ be a quadratic form over a field F. The rank of the matrix A is called the rank of the quadratic form $X'AX$.

Working rule for numerical problems :

We ~~shoud~~ should transform a given symmetric matrix A to a diagonal form by applying congruent operations. Then the application of corresponding columns operations to the unite matrix I_n will be a non singular matrix P ∋

$$P'AP = \text{a diagonal matrix}$$

Practical

Q Determine a non-singular matrix P ∋ $P'AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

~~Interprete the result in terms of quadratic form.~~

Sol^n: We write

$$A = IAI$$

i.e,
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We shall reduce A to a diagonal form by applying congruent operations.

Perfoming the congruent operations we get

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_1 \to C_1 + C_2$

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \\ 5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$

$$\begin{bmatrix} 2 & 1 & 5 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - \frac{1}{2} C_1$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 5 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - \frac{5}{2} R_1$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{25}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{5}{2} & -\frac{5}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - \frac{5}{2} \cancel{R_1} C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{25}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{5}{2} & -\frac{5}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} \\ 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 2 & \frac{1}{2} & -\frac{25}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} \\ 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} \\ 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ we obtain a non singular matrix P,

$$P = \begin{bmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $P'AP = \text{diag}(2, -\frac{1}{2}, -12)$

The quadratic form corresponding to the matrix A is $\cancel{X'AX =}$

$$X'AX = 0.x_1^2 + 0.x_2^2 + 0.x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

$$= 2x_1x_2 + 4x_1x_3 + 6x_2x_3 \quad \text{——} \quad (1)$$

The non singular transformation corresponding to the matrix P is given by $X = PY$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

which is equivalent to

$$\left.\begin{aligned} x_1 &= y_1 \cancel{\#} - y_2/2 - 3y_3 \\ x_2 &= y_1 - y_2/2 - 2y_3 \\ x_3 &= y_3 \end{aligned}\right\} \text{———} (*)$$

The transformation (*) will reduce the quadratic form (1) to the diagonal form

$$2y_1^2 + (-\tfrac{1}{2})y_2^2 + (-12)y_3^2 = 2y_1^2 - \tfrac{1}{2}y_2^2 - 12y_3^2$$

* ~~a~~ Quadratic form is a symmetric matrix.

Q Determine a non singular matrix $P \ni$ $P'AP =$ a diagonal matrix $= \text{diag}\,[\lambda_1, \ldots \lambda_n]$

where
$$A = \begin{bmatrix} 6 & -2 & 2 \\ & 3 & -1 \\ & -1 & -3 \end{bmatrix}$$

Interprete the results in terms of quadratic form.

Sol^n:

$$A = \begin{pmatrix} 6 & -2 & 2 \\ & 3 & -1 \\ & -1 & -3 \end{pmatrix}$$

we can re-write the matrix as

$$A = IAI$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we shall reduce A to a diagonal form by using congruent operation.

$R_2 \to R_2 + \frac{1}{3}R_1$; $C_2 \to C_2 + \frac{1}{2}C_1$, $\cancel{R_3 \to R_3 - \frac{1}{3}R_1}$;

$R_3 \to R_3 - \frac{1}{3}R_1$; $C_3 \to C_3 - \frac{1}{3}C_1$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{7}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{7} & \frac{1}{7} & 0 \end{pmatrix} A \begin{pmatrix} 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix}$$

$R_3 \to R_3 + \frac{1}{7}R_2$; $C_3 \to C_3 + \frac{1}{7}C_2$

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ -\frac{2}{7} & \frac{1}{7} & 1 \end{pmatrix} A \begin{pmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \text{diag}\left(6, \frac{7}{3}, \frac{16}{7}\right) = P'AP$$

where, $P = \begin{pmatrix} 1 & \frac{1}{3} & \frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix}$

$$X'AX = 6X_1^2 + 3X_2^2 + 3X_3^2 - 4X_1X_2 - 2X_2X_3 + 4X_3X_1 = \text{quadratic form}$$

Hence, we can see that the rank of the matrix $\rho(A) = 3 = \rho(X'AX)$

Let, $X = PY$ be the non singular ~~liear~~ linear transformation X to Y.

$$\Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

we have,

$$\left.\begin{aligned} X_1 &= Y_1 + \tfrac{1}{3}Y_2 - \tfrac{2}{7}Y_3 \\ X_2 &= Y_2 + \tfrac{1}{7}Y_3 \\ X_3 &= Y_3 \end{aligned}\right\} \text{——— (1)}$$

This are equivalent solution of X interms of Y. The transformation of equ^n (1) will reduce the quadratic form $X'AX$ coresponding to the given matrix A to a diagonal form as below. so we have

$$X'AX = (PY)'A(PY)$$
$$= Y'P'APY$$
$$= Y'\,\text{diag}\,[6, \tfrac{7}{3}, \tfrac{16}{7}]\,Y$$

$$\Rightarrow [y_1, y_2, y_3] = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 6y_1^2 + \frac{7}{3} y_2^2 + \frac{16}{7} y_3^2$$

Since rank is 3, the quadratic form $X'AX$, so it has been reduced to a form which is sum of three squares namely

$$6y_1^2 + \frac{7}{3} y_2^2 + \frac{16}{7} y_3^2$$

~~1/11/19~~ ~~Signature and~~

Signature & Index of a Quadratic form

Def^n: Let, $y_1^2 + y_2^2 + \cdots y_p^2 - y_{p+1}^2 - \cdots - y_r^2$, be a normal form of a real quadratic form $X'AX$ of rank r. The no. p of positive terms in a normal form $X'AX$ is called the index of the quadratic form.

The excess of the no. of positive terms over the no. of $-ve$ terms in a normal form $X'AX$ i.e, $p-(r-p) = 2p-r$ is called the signature Q.F. & is usually denoted by s.

Thus, $s = 2p - r$

Practical

Q Reduce the following Q.F. to canonical form & find its rank & signature.

$$x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$$

Sol^n: The matrix of the given Q.F.

$A =$ 1, -1, $\cancel{2}\,1/2$; -1, 2; $1/2$

$$A = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$$

we write as

$$A = IAI$$

ie,
$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + C_1$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{9}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - \frac{1}{2} C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{9}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + \frac{1}{2} C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to \frac{1}{\sqrt{2}} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to \frac{1}{\sqrt{2}} C_3$ $\qquad P'AP$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ The linear transformation $X = PY$

$P = (\qquad)$

transforms the given Q.F to a normal from

$y_1^2 + y_2^2 + y_3^2$

$\therefore \rho(A) = 3$

$\therefore$ Signature of the Q.F $= 3 - 0 = 3$

$$\frac{C_3}{\sqrt{2}} \qquad \frac{\frac{1}{2}}{\sqrt{2}} \qquad \frac{1}{2\sqrt{2}} \qquad \frac{C_3}{\sqrt{2}} \qquad \frac{\frac{1}{2}}{\sqrt{2}} \qquad \frac{C_3}{\frac{1}{\sqrt{2}}}$$

page 363 – eg 5.

** Value class of a real quadratic form

~~Def. **Positive definite**: Let, $\phi = X'AX$ be a real Q.F. in n variables $x_1, x_2, --- x_n$ & $\phi = 0$ only if $X = 0$ i.e, $\phi = 0 \Rightarrow x_1 = x_2 = \cdots = x_n = 0$~~

~~**Negative definite**: Let $\phi = X'AX$ be a real~~

21.5 '16

Positive definite:- Let, $\phi = X'AX$ be a real Q.F. in n variables. ~~the form~~ $x_1, x_2, --- x_n$. The form ϕ is said to be +ve definite if $\phi \geq 0$ $\forall$ real values of the variables $x_1, x_2, \cdots x_n$ & $\phi = 0$ only if $X = 0$. i.e, $\phi = 0 \Rightarrow x_1 = x_2 = --- = x_n = 0$. For eg: $x_1^2 + x_2^2 +$ is a +ve definite form in three variables.

Negative definite: Let, $\phi = X'AX$ be a real Q.F in n variables $x_1, x_2, --- x_n$. The form ϕ is said to be −ve definite if $\phi \leq 0$ $\forall$ real values of the variables $x_1, x_2, --- x_n$ & $\phi = 0$ only if $x_1 = x_2 = --- = x_n = 0$.

For eg: $-x_1^2 - x_2^2 - x_3^2$ is a −ve definite form in Three variables.

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16

Positive semi definite : Let, $\phi = X'AX$ be a real Q.F in n variables $x_1, x_2, \cdots x_n$. The form ϕ is said to be a +ve semi definite if $\phi \geq 0$ $\forall$ real values of the variables $x_1, x_2 \cdots x_n$ & $\phi = 0$ for some non zero real vector X i.e, $\phi = 0$ for some real values of the variables $x_1, x_2, \cdots x_n$ not all zero.

For eg : (i) The Q.F. $x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_3 - 2x_2x_3$ is a +ve semi definite because it can be written in the form $(x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 0$ $\forall$ real values of x_1, x_2 & x_3 but is zero for non zero values also.

(ii) The Q.F. $x_1^2 + x_2^2 + 0.x_3^2$ in three variables x_1, x_2, x_3 is a +ve semi definite.

Negative semi definite : Let, $\phi = X'AX$ be a real Q.F. in n variables $x_1, x_2, \cdots x_n$. The form ϕ is said to be -ve semi definite if $\phi \leq 0$ $\forall$ real values of the variable $x_1, x_2 \cdots x_n$ & $\phi = 0$ for some non zero vector X i.e, $\phi = 0$ for some real values of the variables

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$x_1, x_2, \ldots x_n$ not all zero.

For eg: The Q.F $-x_1^2 - x_2^2 - 0.x_3^2$ in three variables x_1, x_2, x_3 is -ve semi definite

Indefinite: Let, $\phi = X'AX$ be a real Q.F in n variables $x_1, x_2, \ldots, x_n$. The form ϕ is said to be indefinite if ϕ takes +ve as well as -ve values of the variables $x_1, x_2, \ldots x_n$.

For eg: The Q.F. $x_1^2 - x_2^2 + x_3^2$ in three ~~va~~ variables is indefinite.

Page: 367, Theorem: 2

Q Prove that the Q.F. ~~since $6x_1^2+3x_2^2+3x_3^2$~~

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$$

Soln: The given matrix of the given Q.F.

$$A = \begin{array}{c|c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ \cancel{x_2} \\ x_3 \end{array} & \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \end{array}$$

The leading principal of minors of A

$A_1 = 6$

$A_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 18 - 4 = 14$

$$A_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) - (-2)(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8 = 32$$

Since the leading pricipal minors of A are all +ve the given Q.F. is +ve definite.

Q Prove that the Q.F.

$6x_1^2 + 49y^2 + 51z^2 - 82yz + 20zx - 4xy$

in three variables in +ve definite.

Q Prove that the Q.F.

$2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$

in three variables is ~~onf~~ indefinite.

Q page: 375 eg: 6